NATIONAL TECHNICAL UNIVERSITY OF ATHENS

SCHOOL OF CIVIL ENGINEERING

INTER-DEPARTMENTAL POSTGRADUATE COURSES PROGRAMMES

«ΔΟΜΟΣΤΑΤΙΚΟΣ ΣΧΕΔΙΑΣΜΟΣ ΚΑΙ ΑΝΑΛΥΣΗ ΚΑΤΑΣΚΕΥΩΝ»

"ANALYSIS AND DESIGN OF EARTHQUAKE RESISTANT STRUCTURES"

GEOMETRICALLY NONLINEAR ANALYSIS OF ELASTIC HOMOGENEOUS ISOTROPIC PRISMATIC BARS : NONLINEAR BENDING TAKING INTO ACCOUNT SHEAR DEFORMATIONS AND NONLINEAR NONUNIFORM TORSION

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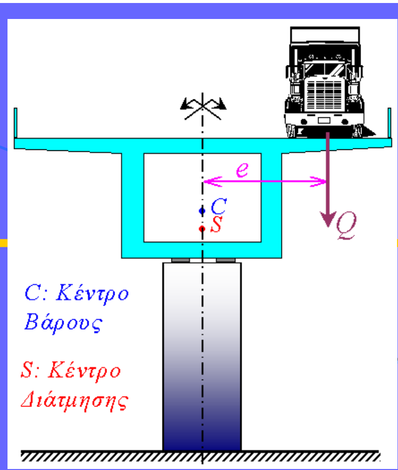
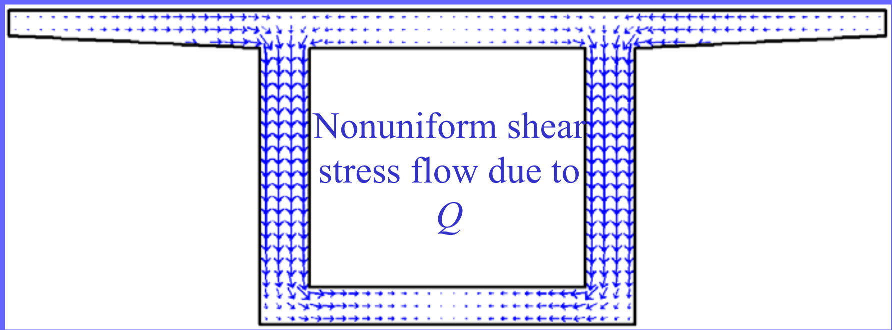
Professor NTUA

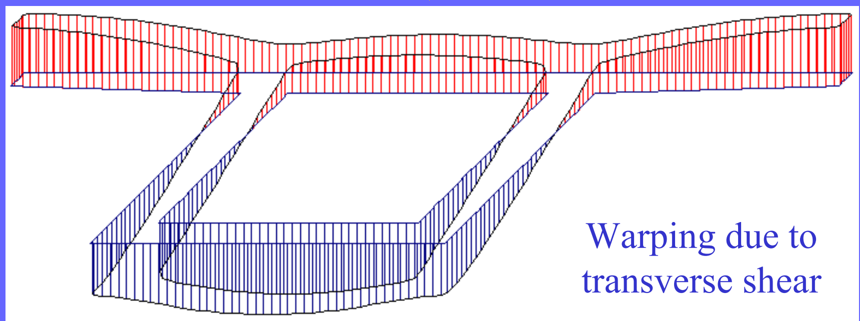
COURSE : APPLIED STRUCTURAL ANALYSIS OF FRAMED AND SHELL STRUCTURES (A1)

Eccentric Transverse Shear Loading *Q*



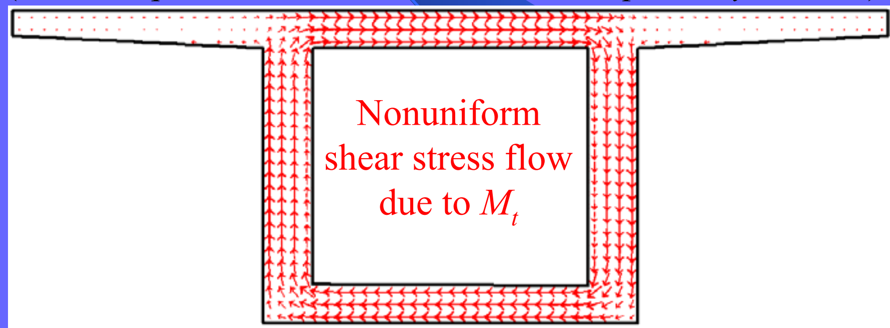
Shear Loading *Q*

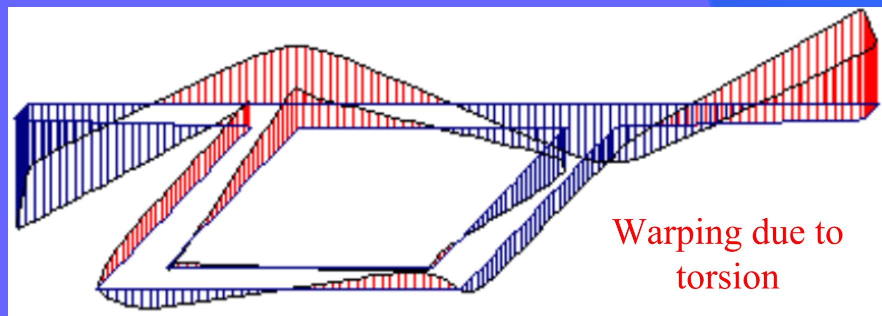




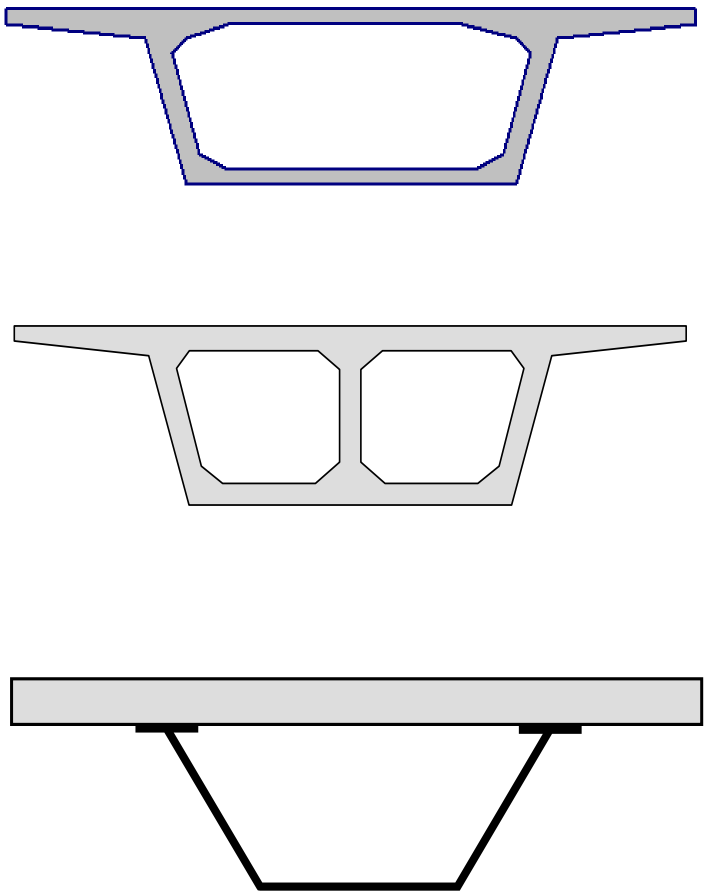
Direct Torsional Loading $M_t = Q \cdot e$)

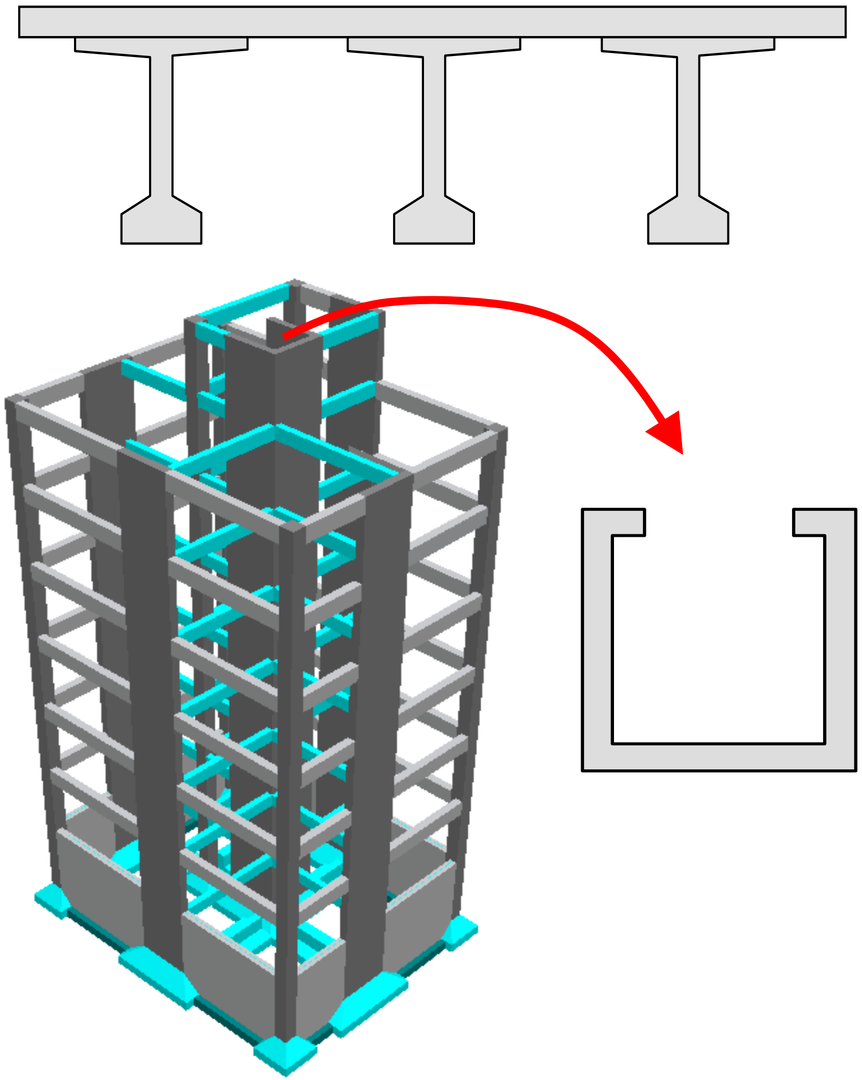
(***Direct***: *Equilibrium Torsion*, ***Indirect***: *Compatibility Torsion*)





CROSS SECTIONS EXHIBITING SMALL AND SIGNIFICANT WARPING





SMALL WARPING

(Closed shaped cross sections)

INTENSE WARPING

(Open shaped cross sections)

Classification of torsion according to longitudinal variation of warping (UNIFORM - NONUNIFORM TORSION)

- **Warping : Free (Not Restrained)**

↓

Uniform Torsion

Linear theory: *Saint–Venant, 1855*

(Are there only shear stresses in case of geometrically nonlinear effects?)

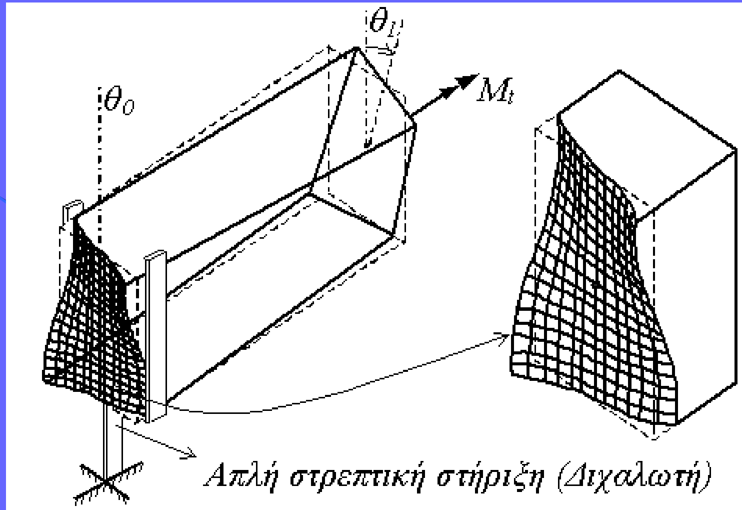
- **Twisting Moment: Variable**
- **Warping (Mt): Restrained**

↓

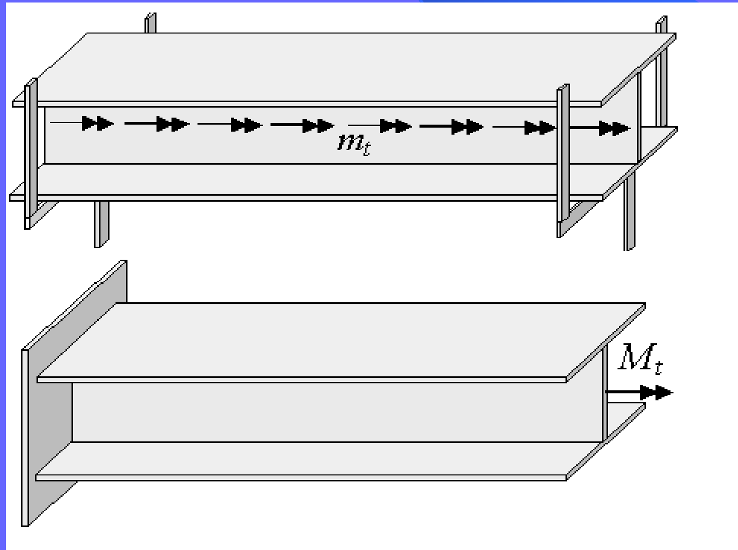
Nonuniform Torsion

Linear theory: *Wagner, 1929*

(Stress field in case of geometrically nonlinear effects?)

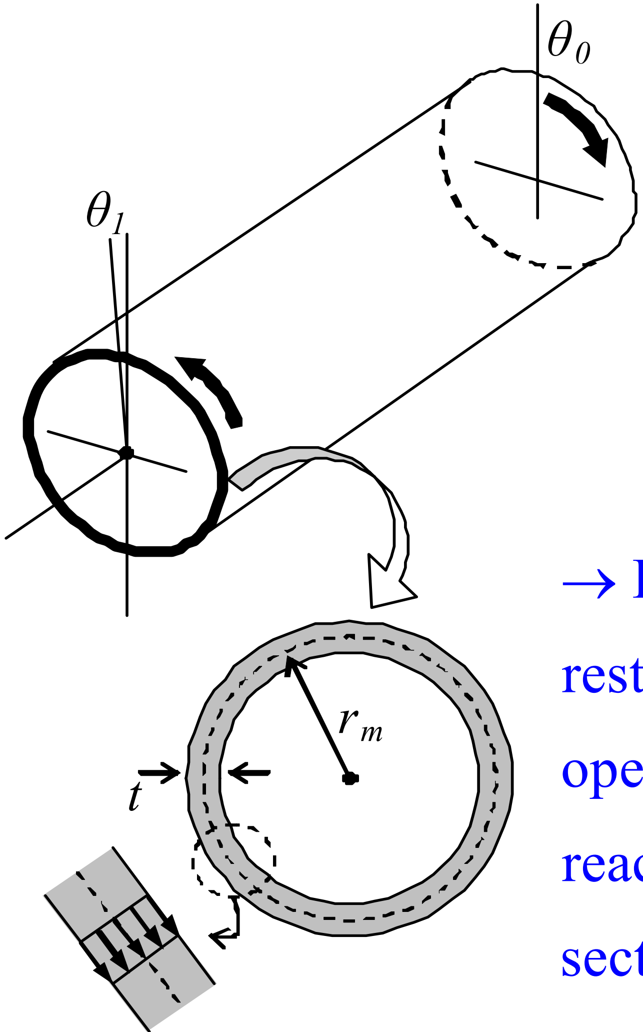


Ομοιόμορφη στρέψη ράβδου ορθογωνικής διατομής



Ανομοιόμορφη στρέψη ράβδων διατομής διπλού "ταυ"

COMPARISON OF TORSIONAL DEFORMATIONS OF THIN WALLED TUBES HAVING CLOSED AND OPEN SHAPED CROSS SECTIONS

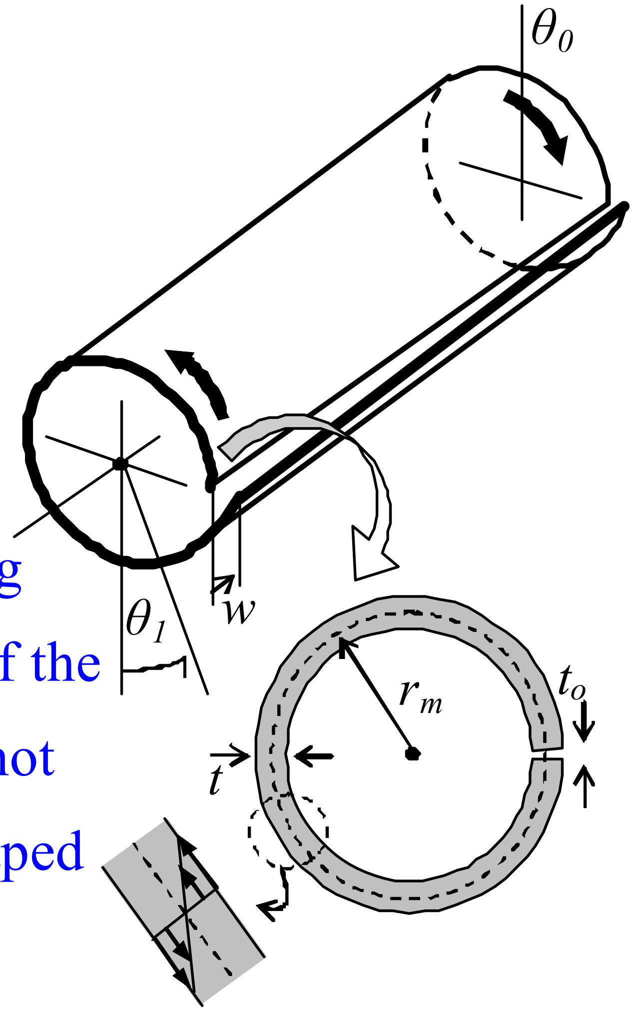


$$r_m = 100mm$$

$$t = 1mm$$

$$I_t^{Close} = 30000 I_t^{Open}$$

→ Even in presence of warping restraints, **torsional rigidity** of the open shaped section bar does not reach the one of the closed shaped section bar

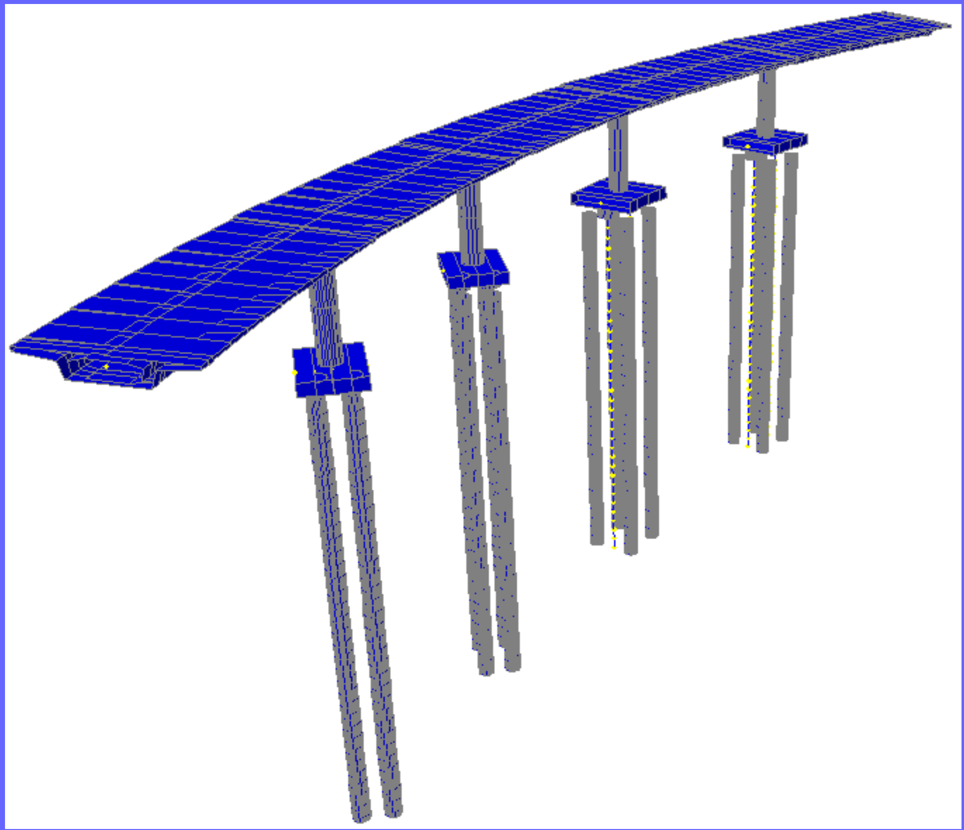


Geometrically nonlinear effects → Members with small torsional rigidity (e.g. with open-shaped sections) are prone to torsional deformations of such magnitudes that it is no longer adequate to treat twisting rotations as **small** even in the linearly elastic regime (large displacement - small strain theory)

CLASSIFICATION OF TORSION AS A STRESS STATE

Direct Torsion

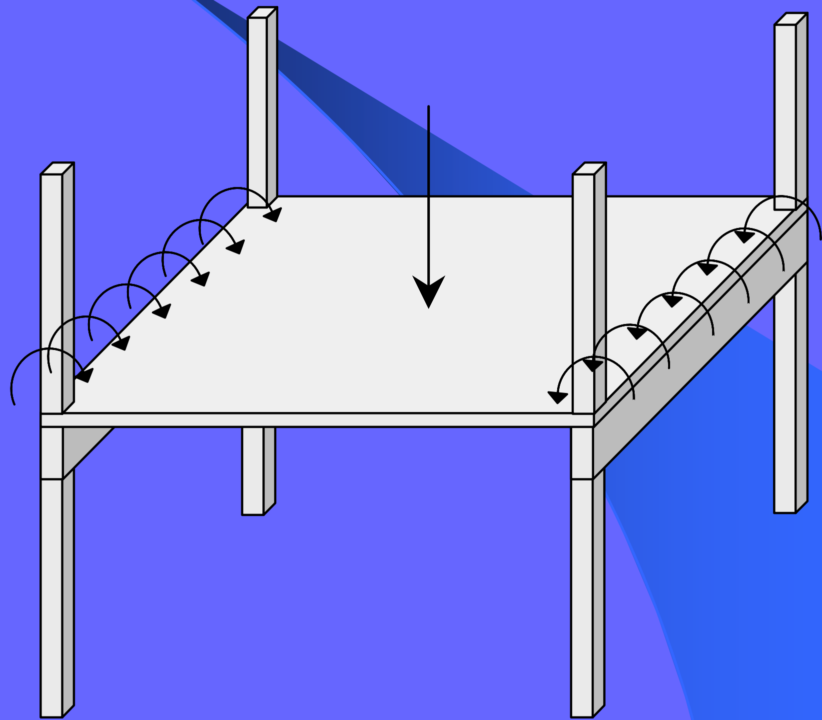
(Equilibrium Torsion)



Bridge deck of box shaped cross section curved in plan → (Permanent) torsional loading due to self-weight

Indirect Torsion

(Compatibility Torsion)



Cracking due to creep and shrinkage effects → Significant reduction of torsional rigidity

Problem description of Nonlinear Nonuniform Torsion

Circular cross sections, uniform torsion *(Young, 1807)* →

- *Circular cross sections (no warping - however axial shortening occurs due to geometrical nonlinearity!)*
- *Uniform torsion (torsional loading is constant along the bar)*

Open shaped thin walled cross sections, nonuniform torsion (*Attard, 1986*) →

- *Valid for thin walled cross sections (Midline employed)*
- *Warping restraints are taken into account (nonuniform torsion theory)*
- *Arbitrary torsional loading conditions (nonuniform torsion theory)*
- *Reliability: Depends on thickness of shell elements comprising the beam*

Arbitrarily shaped cross sections, nonuniform torsion (*Sapountzakis and Tsipiras, 2010*) →

- *Valid for arbitrarily shaped cross sections (Thick or Thin walled)*
- *Warping restraints are taken into account*
- *Arbitrary torsional loading conditions (nonuniform torsion theory)*
- *BVPs formulated employing theory of 3D elasticity*
- *Numerical solution of BVPs*

ASSUMPTIONS OF ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

- The bar is straight.
- The bar is prismatic.
- Distortional deformations of the cross section are not allowed (cross sectional shape is not altered during deformation ($\gamma_{23}=0$, *distortion neglected*).
- The bar is subjected to torsional loading along its longitudinal axis and bar ends exclusively. Axial and flexural boundary conditions are not arbitrary.
- The bar may twist freely. None axis of twist is imposed due to construction requirements
- Secondary torsional moment deformation effect (taking into account warping shear stresses in the global equilibrium of the bar) is neglected (This effect is important in bars of closed shaped cross sections (Massonnet, 1983) and in short bars).
- Flexural displacements of the cross section do not induce transverse shear deformations (analogous with the Bernoulli-Euler assumption of flexural loading conditions).
- Bending rotations of the cross section are assumed to be small to moderate large. Axial shortening and warping of the bar are assumed to be small.
- The material of the bar is homogeneous, isotropic, continuous (no cracking) and linearly elastic: Constitutive relations of linear elasticity are valid (small strain theory).
- The distribution of stresses at the bar ends is such so that all the aforementioned assumptions are valid.

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Consider a prismatic bar of length *l* with an arbitrarily shaped constant cross-section occupying the *y, z* plane (area *A*)

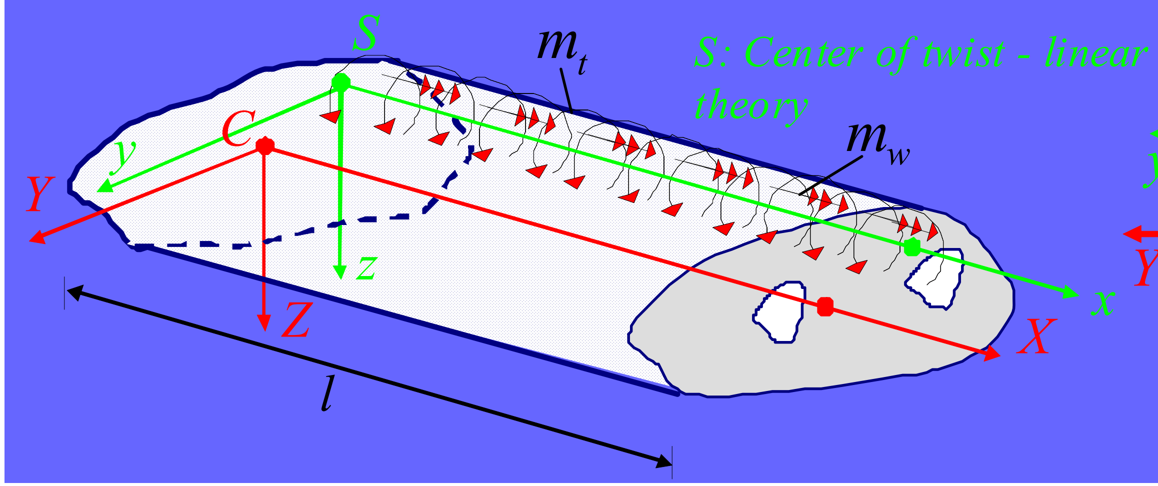
Material: homogeneous, isotropic, linearly elastic with modulus of elasticity *E*, shear modulus *G*

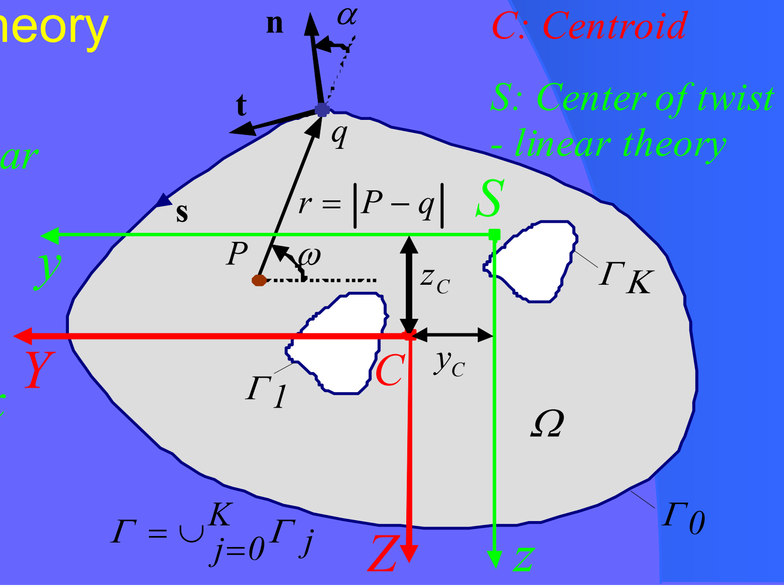
- The bar is subjected to arbitrarily distributed or concentrated conservative twisting $m_t = m_t(x)$ and warping $m_w = m_w(x)$ moments

Employing a principal center of twist coordinate system Sxyz, the transverse displacement components valid for large rotations $\theta_x(x)$ are derived as

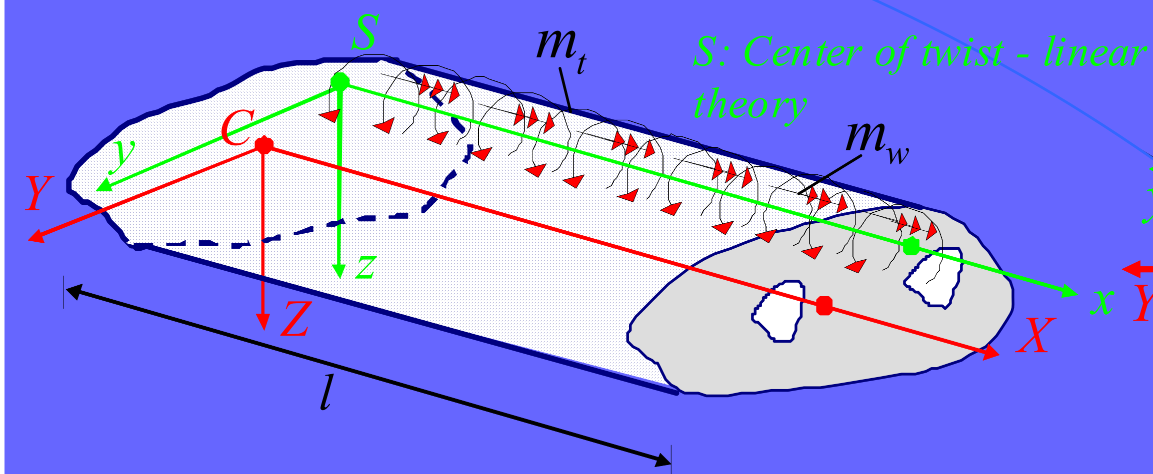
$$v = v_S(x) - z \cdot sin\,\theta_x(x) - y \cdot (1 - cos\,\theta_x(x)) \qquad w = w_S(x) + y \cdot sin\,\theta_x(x) - z \cdot (1 - cos\,\theta_x(x))$$

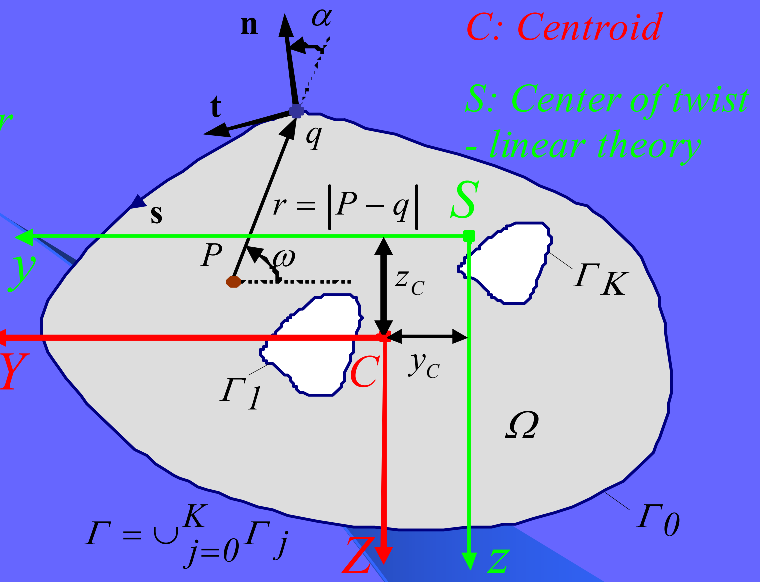
$v_S(x), w_S(x)$: transverse displacements of the cross-section as a rigid body with respect to the center of twist S of linear theory





ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION





$$v = v_S(x) - z \cdot \sin\theta_x(x) - y \cdot (1 - \cos\theta_x(x)) \quad w = w_S(x) + y \cdot \sin\theta_x(x) - z \cdot (1 - \cos\theta_x(x))$$

$v_S(x), w_S(x)$: transverse displacements of the cross section as a rigid body with respect to the center of twist S of linear theory

$\rightarrow$ Valid for arbitrarily large twisting rotations θ_x

$\rightarrow$ $v_S(x) = w_S(x) = 0$ only for doubly symmetric cross sections. Transverse displacements of the cross section must arise in order to have zero flexural load!

$\rightarrow$ There is not any axis of twist for monosymmetric or asymmetric sections (more on this later...). All fibers are displaced transversely (unlike the linear theory)

$\rightarrow$ Any other coordinate system other than Sxyz could be used, although the employed one simplifies the expressions of the theory

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Assuming small bending rotations and vanishing secondary torsional moment deformation effect (STMDE), the longitudinal displacement component is given as:

$$u = u_m(x) + \theta_Y(x)\cdot(z - z_C) - \theta_Z(x)\cdot(y - y_C) + \theta'_x(x)\cdot\phi_S^P(y,z) + \phi_S^S(x,y,z)$$

$u_m(x)$: "average" axial displacement of the cross-section $\rightarrow$ axial shortening of the bar must arise in order to have zero axial load (Young, 1807)! Fibers become helices in space (even without warping effects) whereas linear theory assumes that fibers remain straight (Attard, 1986)

$\theta_Y(x), \theta_Z(x)$: Angles of rotation due to bending of the cross-section with respect to its centroid C

$\rightarrow$ $\theta_Y(x) = \theta_Z(x) = 0$ only for doubly symmetric cross sections

$\theta'_x(x)$: The angle of twist per unit length (torsional curvature)

$\rightarrow$ primary warping analogous to $\theta'_x(x)$: STMDE neglected

$\phi_S^P(y,z), \phi_S^S(x,y,z)$: primary and secondary warping functions with respect to the shear center S (related with torsional warping)

$\rightarrow$ Warping shear stresses are calculated (a posteriori), however their effect on global equilibrium of the bar (STMDE) is neglected

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Center of Twist (S) - Taken as in linear theory!

$\bar{x}_2^S, \bar{x}_3^S$: Point with respect to which the cross sections rotate (*no transverse displacements*) (or point where rotation causes no axial and bending stress resultants

$\tau_{12}^P, \tau_{13}^P, I_t$: Independent of the center of twist (St. Venant could not calculate the position of the center of twist!)

$u_1^P, \tau_{12}^S, \tau_{13}^S, \tau_{11}^w, C_S$: Dependent of the center of twist

$$\varphi_S^P(\tilde{x}_2, \tilde{x}_3) = \phi_O^P(\bar{x}_2, \bar{x}_3) - \bar{x}_2 \bar{x}_3^S + \bar{x}_3 \bar{x}_2^S + \bar{c}$$

$$\nabla^2 \varphi_O^P = 0 \,,\ \Omega \qquad \frac{\partial \varphi_O^P}{\partial n} = \bar{x}_3 \cdot n_2 - \bar{x}_2 \cdot n_3 \,,\ \Gamma$$

- **Method of equilibrium:**

Under any coordinate system $N = M_2 = M_3 = 0$ due to warping normal stresses

- **Energy Method:**

Minimization of Strain Energy due to warping normal stresses

$$\frac{\partial C_M}{\partial \bar{x}_2} = \frac{\partial C_M}{\partial \bar{x}_3} = \frac{\partial C_M}{\partial \bar{c}} = 0$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Center of Twist (S) of linear theory

$$\bar{S}_2\, \bar{x}_2^S - \bar{S}_3\, \bar{x}_3^S + A\, \bar{c} = -\bar{R}_S^P$$

$$\bar{I}_{22}\, \bar{x}_2^S + \bar{I}_{23}\, \bar{x}_3^S + \bar{S}_2\, \bar{c} = -\bar{R}_2^P$$

$$\bar{I}_{23}\, \bar{x}_2^S + \bar{I}_{33}\, \bar{x}_3^S - \bar{S}_3\, \bar{c} = \bar{R}_3^P$$

where:

$$A = \int_\Omega d\Omega \qquad \bar{S}_2 = \int_\Omega \bar{x}_3\, d\Omega \qquad \bar{S}_3 = \int_\Omega \bar{x}_2\, d\Omega$$

$$\bar{I}_{22} = \int_\Omega \bar{x}_3{}^2 d\Omega \qquad \bar{I}_{33} = \int_\Omega \bar{x}_2{}^2 d\Omega \qquad \bar{I}_{23} = -\int_\Omega \bar{x}_2 \bar{x}_3\, d\Omega$$

$$\bar{R}_S^P = \int_\Omega \varphi_O^P\, d\Omega \qquad \bar{R}_2^P = \int_\Omega \bar{x}_3\, \varphi_O^P\, d\Omega \qquad \bar{R}_3^P = \int_\Omega \bar{x}_2\, \varphi_O^P\, d\Omega$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Non-vanishing Green shear strains valid for moderate – large bending rotations, small axial shortening and warping (suitable for geometrically nonlinear analysis):

$$\gamma_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \left(\cancel{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right) \qquad \gamma_{xz} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + \left(\cancel{\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \right)$$

$$\gamma_{xy} = -\theta_Z + \left(v_S' \cos\theta_x + w_S' \sin\theta_x \right) + \theta_x' \underbrace{\left(\left(\frac{\partial \phi_S^P}{\partial y} \right) - z \right)}_{\text{primary}} + \underbrace{\left(\frac{\partial \phi_S^S}{\partial y} \right)}_{\text{secondary}}$$

$$\gamma_{xz} = \theta_Y - \left(v_S' \sin\theta_x - w_S' \cos\theta_x \right) + \theta_x' \underbrace{\left(\left(\frac{\partial \phi_S^P}{\partial z} \right) + y \right)}_{\text{primary}} + \underbrace{\left(\frac{\partial \phi_S^S}{\partial z} \right)}_{\text{secondary}}$$

Employing the "Bernoulli - Euler" assumption (vanishing transverse shear deformation) the angles of rotation due to bending are obtained as

$$\theta_Y(x) = v_S{}' \cdot \sin\theta_x - w_S{}' \cdot \cos\theta_x \qquad \theta_Z(x) = v_S{}' \cdot \cos\theta_x + w_S{}' \cdot \sin\theta_x$$

$\rightarrow$ Same result by assuming that the cross section is normal to the deformed axis Sx

$\rightarrow$ Same result through thin-walled beam theory by assuming vanishing shear strains at the midline of the shell elements comprising the bar (Attard, 1986)

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

The non-vanishing Green's strain components are evaluated: $\left(\varepsilon_{yy}=\varepsilon_{zz}=\gamma_{yz}=0\right)$

$$\left(\varepsilon_{xx}=\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\right)$$

$$\varepsilon_{xx}=u_m{}'+\kappa_Y\cdot(z-z_C)-\kappa_Z\cdot(y-y_C)+\theta''_x\cdot\phi_S^P(y,z)-\theta'_x\cdot(y_C\cdot\theta_Y+z_C\cdot\theta_Z)+$$
$$+\frac{1}{2}\left[\left(v_S{}'\right)^2+\left(w_S{}'\right)^2+\left(y^2+z^2\right)\cdot(\theta'_x)^2\right]$$

$$\gamma_{xy}=\theta'_x\cdot\left(\frac{\partial\phi_S^P}{\partial y}-z\right)+\frac{\partial\phi_S^S}{\partial y}\qquad\gamma_{xz}=\theta'_x\cdot\left(\frac{\partial\phi_S^P}{\partial z}+y\right)+\frac{\partial\phi_S^S}{\partial z}$$

$\rightarrow$ Secondary warping function ϕ_S^S has been ignored in the normal strain component

$\frac{1}{2}\left(y^2+z^2\right)\cdot(\theta'_x)^2$: Second order geometrically nonlinear term of ε_{xx}, "Wagner effect"

$\rightarrow$Responsible for the axial shortening in doubly symmetric cross section bars

κ_Y,κ_Z: curvature components (due to bending)

$$\kappa_Y(x)=v_S{}''(x)\cdot\sin\theta_x-w_S{}''(x)\cdot\cos\theta_x$$

$$\kappa_Z(x)=v_S{}''(x)\cdot\cos\theta_x+w_S{}''(x)\cdot\sin\theta_x$$

$\rightarrow$Responsible for flexural deformations in monosymmetric and asymmetric cross section bars

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

❖ By considering strains to be small, the generalized Hooke's stress-strains relations are employed to resolve the work contributing second Piola – Kirchhoff stress components (suitable for geometrically nonlinear analysis - work conjugate with Green's strain components)

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \Rightarrow$$

Term related to nonuniform warping

$$S_{xx} = E\Big\{u'_m + \kappa_Y\left(z - z_C\right) - \kappa_Z\left(y - y_C\right) + \theta''_x \phi_S^P - \theta'_x\left(y_C\theta_Y + z_C\theta_Z\right) + \frac{1}{2}\left[\left(v'_S\right)^2 + \left(w'_S\right)^2 + \left(y^2 + z^2\right)\left(\theta'_x\right)^2\right]\Big\}$$

Normal stresses are not caused solely due to (nonuniform) warping (unlike the linear theory)

$$S_{xy} = G\left[\theta'_x\left(\frac{\partial \phi_S^P}{\partial y} - z\right) + \frac{\partial \phi_S^S}{\partial y}\right]$$

$$S_{xz} = G\left[\theta'_x\left(\frac{\partial \phi_S^P}{\partial z} + y\right) + \frac{\partial \phi_S^S}{\partial z}\right]$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

For the nonlinear torsion problem, it has been proved (through a variational formulation) that the 2nd Piola – Kirchhoff stress components may be introduced into the longitudinal differential equation of equilibrium (Washizu, 1975)

$\rightarrow$ This differential equation is used to resolve the unknown warping functions ϕ_S^P, ϕ_S^S (differential equilibrium equations at the transverse directions are not satisfied as in linear theory):

$$\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}+\frac{\partial S_{xz}}{\partial z}=0 \text{ in region } \Omega \qquad S_{xy}n_y+S_{xz}n_z=0 \text{ on boundary } \Gamma$$

$\rightarrow$ Decomposition of stresses into primary and secondary parts (as in linear theory)

Two boundary value problems (Neumann type boundary conditions) are obtained after the decomposition of shear stresses into primary and secondary parts:

$$\nabla^2\phi_S^P=0 \text{ in } \Omega \qquad \frac{\partial\phi_S^P}{\partial n}=z\cdot n_y-y\cdot n_z \text{ on } \Gamma$$

$$\nabla^2\phi_S^S=-\frac{E}{G}\Big\{u_m''+\kappa_Y'\left(z-z_C\right)-\kappa_Z'\left(y-y_C\right)+\theta_x'''\phi_S^P$$

$$-\theta_x''\left(y_C\theta_Y+z_C\theta_Z\right)-\theta_x'\left(y_C\theta_Y'+z_C\theta_Z'\right) \qquad \frac{\partial\phi_S^S}{\partial n}=\frac{t_x}{G} \text{ on } \Gamma$$

$$+v_S'v_S''+w_S'w_S''+\left(y^2+z^2\right)\theta_x'\theta_x''\Big\} \text{ in } \Omega$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

- Definition of stress resultants with respect to the deformed configuration

$$N = \int_{\Omega} S_{xx} d\Omega \qquad M_Y = \int_{\Omega} S_{xx}\left(z - z_C\right) d\Omega \qquad M_Z = -\int_{\Omega} S_{xx}\left(y - y_C\right) d\Omega$$

$M_t^P = \int_{\Omega}\left[S_{xy}^P \cdot \left(\frac{\partial \phi_S^P}{\partial y} - z\right) + S_{xz}^P \cdot \left(\frac{\partial \phi_S^P}{\partial z} + y\right)\right] d\Omega$: Primary twisting moment (St. Venant)

$M_w = -\int_{\Omega} S_{xx} \phi_S^P d\Omega$: Warping moment

The rotations of the infinitesimal surfaces comprising the cross section occuring during deformation are taken into account through these definitions

$$N = EA \cdot \left[u_m' + \frac{1}{2} \cdot \left(\left(v_S'\right)^2 + \left(w_S'\right)^2 + \frac{I_P}{A} \cdot \left(\theta_x'\right)^2\right) - \theta_x' \cdot \left(y_C \theta_Y + z_C \theta_Z\right)\right]$$

$$M_Y = EI_{YY} \cdot \left[\kappa_Y + \beta_2 \left(\theta_x'\right)^2\right] \qquad M_Z = EI_{ZZ} \cdot \left[\kappa_Z - \beta_1 \left(\theta_x'\right)^2\right]$$

where: $A = \int_{\Omega} d\Omega \quad I_P = \int_{\Omega}\left(y^2 + z^2\right) d\Omega \quad I_{YY} = \int_{\Omega}\left(z - z_C\right)^2 d\Omega \quad I_{ZZ} = \int_{\Omega}\left(y - y_C\right)^2 d\Omega$

$\beta_1 = \frac{1}{2I_{ZZ}} \int_{\Omega}\left(y^2 + z^2\right)\left(y - y_C\right) d\Omega, \; \beta_2 = \frac{1}{2I_{YY}} \int_{\Omega}\left(y^2 + z^2\right)\left(z - z_C\right) d\Omega$:

Terms associated with bending due to (geometrically) nonlinear torsion ($\beta_1 = \beta_2 = 0$ for doubly symmetric cross-sections)

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

$$M_t^P = GI_t \cdot \theta'_x \qquad M_w = -EC_S \cdot \left[\theta''_x + \frac{U_w}{2C_S} \cdot (\theta'_x)^2 \right]$$

$I_t = \int_\Omega \left(y^2 + z^2 + y \cdot \frac{\partial \phi_S^P}{\partial z} - z \cdot \frac{\partial \phi_S^P}{\partial y} \right) d\Omega$: St. Venant's torsion constant

$C_S = \int_\Omega \left(\phi_S^P \right)^2 d\Omega$: Warping constant

$U_w = \int_\Omega \phi_S^P \cdot \left(y^2 + z^2 \right) d\Omega$: Term associated with the modification of the warping moment due to (geometrically) nonlinear torsion (Attard, 1986)

- Principle of virtual work under a total Lagrangian formulation (importance of variational methods in nonlinear problems: Attard, 1986)

$$\int_V \left(S_{xx} \cdot \delta\varepsilon_{xx} + S_{xy} \cdot \delta\gamma_{xy} + S_{xz} \cdot \delta\gamma_{xz} \right) dV = \int_F \left(t_x \cdot \delta u + t_y \cdot \delta v + t_z \cdot \delta w \right) dA$$

➢ Governing equation of torque equilibrium of the bar (for global moment, transverse shear and axial force equilibrium equations, see Attard, 1986: Analysis of flexural, torsional, flexural-torsional and lateral-torsional buckling of bars)

$$-Ny_C\theta_Z\theta'_x + Nz_C\theta_Y\theta'_x - M_Z\kappa_Y + M_Y\kappa_Z$$
$$-\frac{d}{dx}\left[M_t^P + \frac{1}{2}EI_n(\theta'_x)^3 + \Psi\theta'_x - Ny_C\theta_Y - Nz_C\theta_Z \right] - \frac{d^2M_w}{dx^2} = m_t(x) + \frac{d}{dx}\left[m_w(x) \right]$$

➢ Corresponding boundary conditions at the bar's ends

$$\left[(M_w)' + M_t^P + \frac{1}{2}EI_n(\theta'_x)^3 + \Psi\theta'_x - Ny_C\theta_Y - Nz_C\theta_Z + m_w - M_{\bar{t}} \right]\delta\theta_x = 0 \text{ (1)} \quad \left(-M_w + M_{\bar{w}} \right)\delta\theta'_x = 0 \text{ (2)}$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

For pure torsional loading the axial and bending stress resultants vanish, thus the governing torque equilibrium equation becomes:

$$EC_S\theta_x''''-\frac{3}{2}EI_{n2}\left(\theta_x'\right)^2\theta_x''-GI_t\theta_x''=m_t\left(x\right)+\frac{d}{dx}\left[m_w\left(x\right)\right]$$

with the most general boundary conditions at the bar ends:

$a_1M_t+\alpha_2\theta_x=\alpha_3$ (1) $\beta_1M_w+\beta_2\theta_x'=\beta_3$ (2)

$\frac{3}{2}EI_{n2}\left(\theta_x'\right)^2\theta_x''$: Nonlinear term accounting for large rotations

$$I_{n2}=\int_\Omega\left(y^2+z^2\right)^2d\Omega-\frac{{I_P}^2}{A}-4{\beta_1}^2I_{ZZ}-4{\beta_2}^2I_{YY}$$

$m_t\left(x\right),m_w\left(x\right)$: externally applied conservative twisting and warping moments

$$m_t\left(x\right)=\int_\Gamma t_y\left(-z\cos\theta_x-y\sin\theta_x\right)+t_z\left(y\cos\theta_x-z\sin\theta_x\right)ds \qquad m_w\left(x\right)=-\int_\Gamma t_x\phi_S^P\,ds$$

$\rightarrow$ Geometrical nonlinearity alters the expressions of external loading

$$M_t=-EC_S\theta_x'''+GI_t\theta_x'+\frac{1}{2}EI_{n2}\left(\theta_x'\right)^3+m_w \qquad M_w=-EC_S\left[\theta_x''+\frac{U_w}{2C_S}\left(\theta_x'\right)^2\right]$$

α_i,β_i : constants specified at the bar ends (e.g. fully clamped edge: $a_2=\beta_2=1,\ a_1=a_3=\beta_1=\beta_3=0)\rightarrow$ Any type of boundary conditions can be applied by appropriately specifying α_i,β_i

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

$$EC_S\theta_x'''' - \frac{3}{2}EI_{n2}\left(\theta_x'\right)^2\theta_x'' - GI_t\theta_x'' = m_t\left(x\right) + \frac{d}{dx}\left[m_w\left(x\right)\right],\ x \in \left(0,l\right)$$

$$a_1M_t + \alpha_2\theta_x = \alpha_3 \quad (1) \qquad \beta_1M_w + \beta_2\theta_x' = \beta_3 \quad (2)$$

$$M_t = -EC_S\theta_x''' + GI_t\theta_x' + \frac{1}{2}EI_{n2}\left(\theta_x'\right)^3 + m_w \qquad M_w = -EC_S\left[\theta_x'' + \frac{U_w}{2C_S}\left(\theta_x'\right)^2\right]$$

$\rightarrow$ For monosymmetric (or doubly symmetric) cross section bars, $U_w = 0 \Rightarrow$ In case of uniform torsion and unrestrained warping, θ_x'=constant and

$$M_t = GI_t\theta_x' + \frac{1}{2}EI_{n2}\left(\theta_x'\right)^3$$

$\rightarrow$ For asymmetric cross section bars, $U_w \neq 0 \Rightarrow$ In case of uniform torsion and unrestrained warping, $\theta_x' \neq$ constant, unlike the linear theory!

$\rightarrow$ Geometrical nonlinearity always results in increase of torsional rigidity (in absence of axial and flexural loading) since straight lines become helices

$\rightarrow$ Torsion members become fully plastic: Use of plastic methods of strength design is always possible

$\rightarrow$ Elastic lateral torsional postbuckling behaviour is imperfection insensitive

$\rightarrow$ Lateral buckling strength of a beam bent about its strong axis cannot be less than its weak axis in-plane strengh

Trahair, 2005

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

$\rightarrow$ After the resolution of the angle of twist θ_x the rest of the kinematical components can be obtained as:

$$\left.\begin{aligned}\kappa_Y(x) &= v_S''(x)\cdot\sin\theta_x - w_S''(x)\cdot\cos\theta_x \\ \kappa_Z(x) &= v_S''(x)\cdot\cos\theta_x + w_S''(x)\cdot\sin\theta_x\end{aligned}\right\} \xrightarrow[0=M_Z=EI_{ZZ}\cdot\left[\kappa_Z-\beta_1(\theta_x')^2\right]]{0=M_Y=EI_{YY}\cdot\left[\kappa_Y+\beta_2(\theta_x')^2\right]}$$

$$\left\{\begin{aligned}v_s'' &= -(\theta_x')^2\cdot(\beta_2\cdot\sin\theta_x - \beta_1\cdot\cos\theta_x) \\ w_s'' &= (\theta_x')^2\cdot(\beta_2\cdot\cos\theta_x + \beta_1\cdot\sin\theta_x)\end{aligned}\right\}$$

(subsequent numerical integration)

$$N = EA\cdot\left[u_m' + \frac{1}{2}\cdot\left(\left(v_S'\right)^2 + \left(w_S'\right)^2 + \frac{I_P}{A}\cdot(\theta_x')^2\right) - \theta_x'\cdot\left(y_C\theta_Y + z_C\theta_Z\right)\right] \rightarrow$$

$N = 0$: Equilibrium of axial forces

$$u_m' = -\frac{1}{2}\cdot\left[\left(v_s'\right)^2 + \left(w_s'\right)^2 + \frac{I_P}{A}\cdot(\theta_x')^2\right] + \theta_x'\cdot\left(y_C\theta_Y + z_C\theta_Z\right)$$

(subsequent numerical integration)

$\rightarrow$ Axial shortening always arises (the term $\frac{1}{2}\frac{I_P}{A}(\theta_x')^2$ does not vanish even in doubly symmetric cross section bars) in order to have zero axial force

$\rightarrow$ Monosymmetric or asymmetric cross section bars exhibit transverse deflections: There is not any undisplaced axis of twist unlike linear theory

$\rightarrow$ Axial and flexural boundary conditions cannot be arbitrary in order to have zero axial, bending and transverse shear loading

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

The BVP yielding the secondary warping fuction is simplified as

$$\nabla^2 \phi_S^S = -\frac{E}{G}\left\{\left[\left(y^2+z^2\right)-\frac{I_P}{A}\right]\theta_x'\theta_x'' + \kappa_Y'\left(z-z_C\right)-\kappa_Z'\left(y-y_C\right)+\theta_x'''\phi_S^P\right\} \text{ in } \Omega \qquad \frac{\partial \phi_S^S}{\partial n}=\frac{t_x}{G} \text{ on } \Gamma$$

$\rightarrow$ Secondary shear stresses arise even in case of a monosymmetric cross section bar under uniform torsion

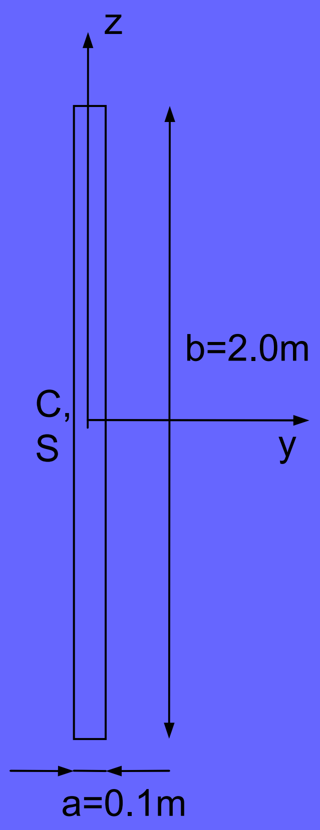
$$\nabla^2 \phi_S^S = -\frac{E}{G}\left[\kappa_Y'\left(z-z_C\right)-\kappa_Z'\left(y-y_C\right)\right] \text{ in } \Omega \qquad \frac{\partial \phi_S^S}{\partial n}=\frac{t_x}{G} \text{ on } \Gamma$$

$\rightarrow$ Secondary shear stresses vanish only in case of a doubly symmetric cross section bar under uniform torsion

$$\phi_S^S = 0$$

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

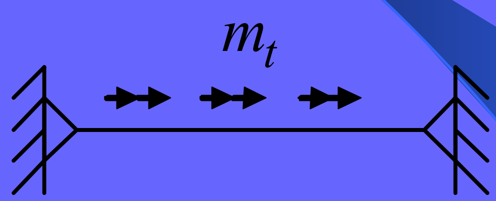
Example 1 Narrow rectangle Cross-section (doubly symmetric)



Simply supported torsion member (free warping at both ends)

Uniformly applied torsional loading along the bar

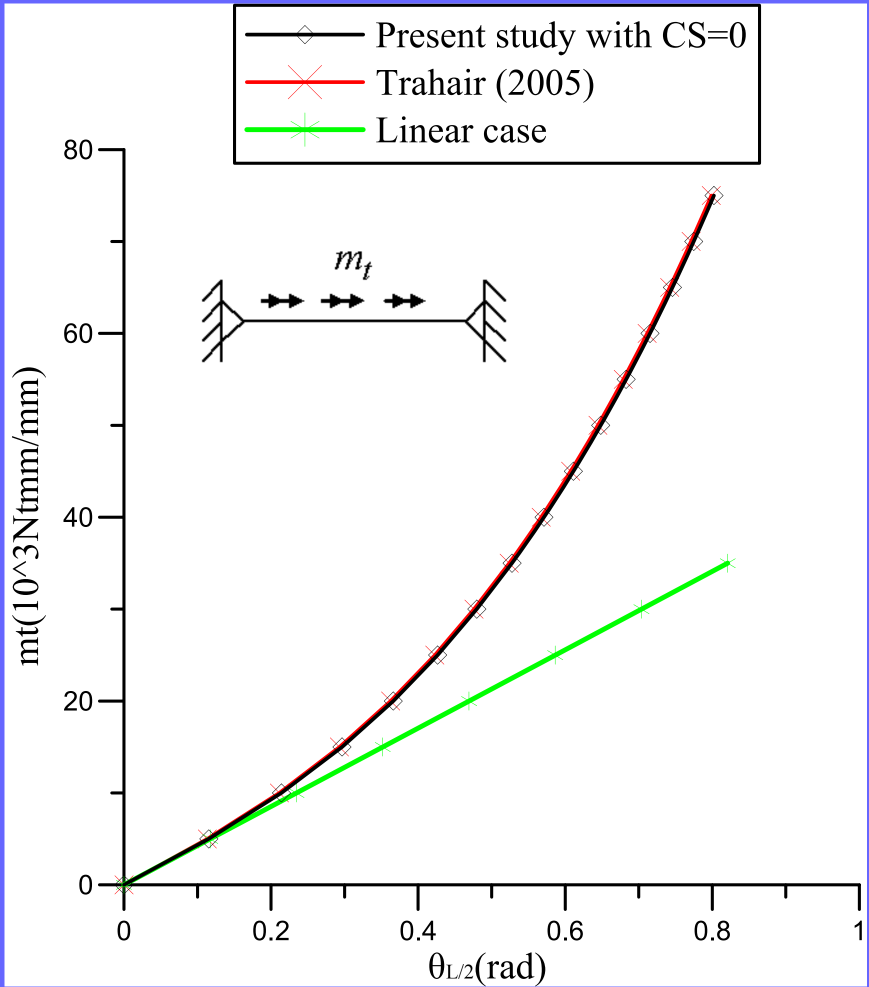
$E = 200000\ MPa$
$G = 80000\ MPa$
$L = 1,0\ m$



	Present study	Trahair (2005)
$I_t\left(\times 10^4\ mm^4\right)$	6,6488	6,6667
$C_S\left(\times 10^7\ mm^6\right)$	5,499	0
$I_{n2}\left(\times 10^{10}\ mm^6\right)$	1,7778	1,7778

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 1 Narrow rectangle Cross-section (doubly symmetric)



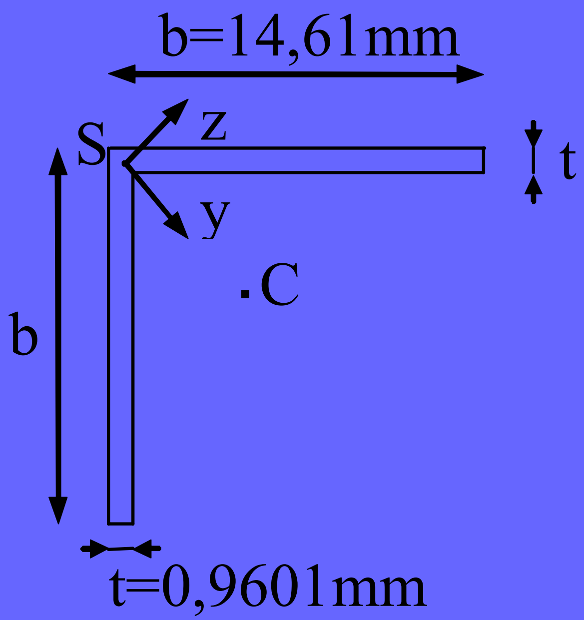
- Stiffening of the bar due to the geometrical nonlinearity

$m_t\left(10^3\,kNmm/mm\right)$	Present study - Angle of twist θ_x	
	$C_S = 5.499\times 10^7\,mm^6$	$C_S = 0$
10	0,2114	0,2142
20	0,3618	0,3661
30	0,4741	0,4798
40	0,5643	0,5712
50	0,6403	0,6483
60	0,7063	0,7153
70	0,7650	0,7748

- Only minor discrepancy of the results if nonuniform warping is neglected (due to the shape of the cross section)

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 2 Angle section (monosymmetric)



$E = 89660\ MPa$

$G = 31130\ MPa$

$L = 177,8\ mm$

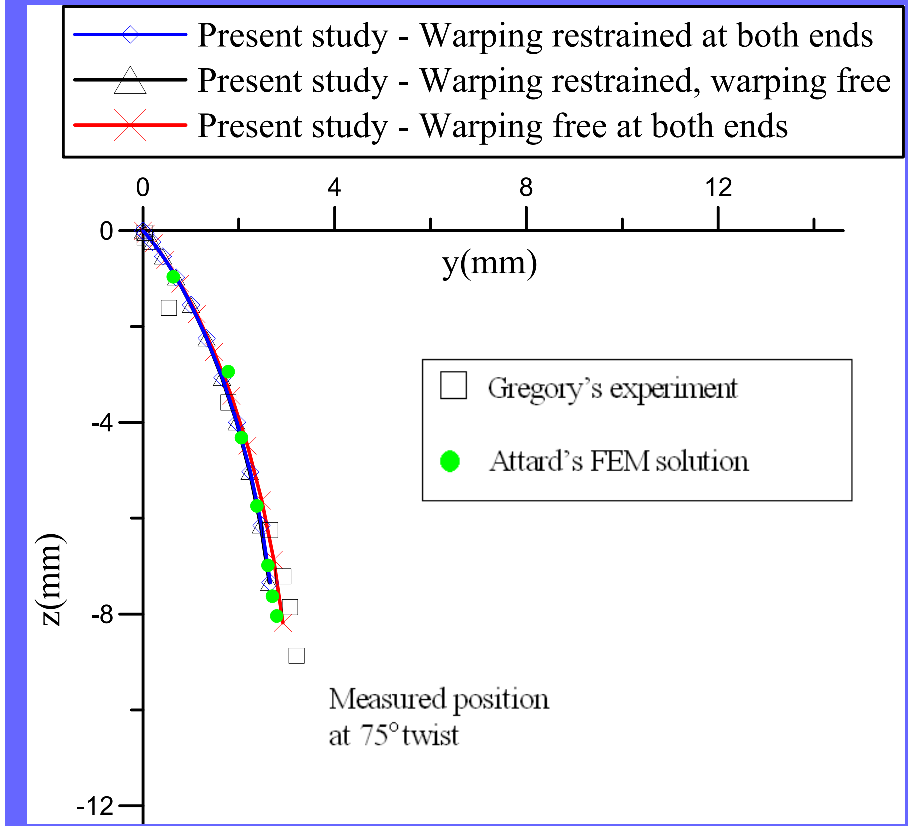
<u>Concentrated torque</u> M_t applied at the free end. Three boundary conditions are investigated:

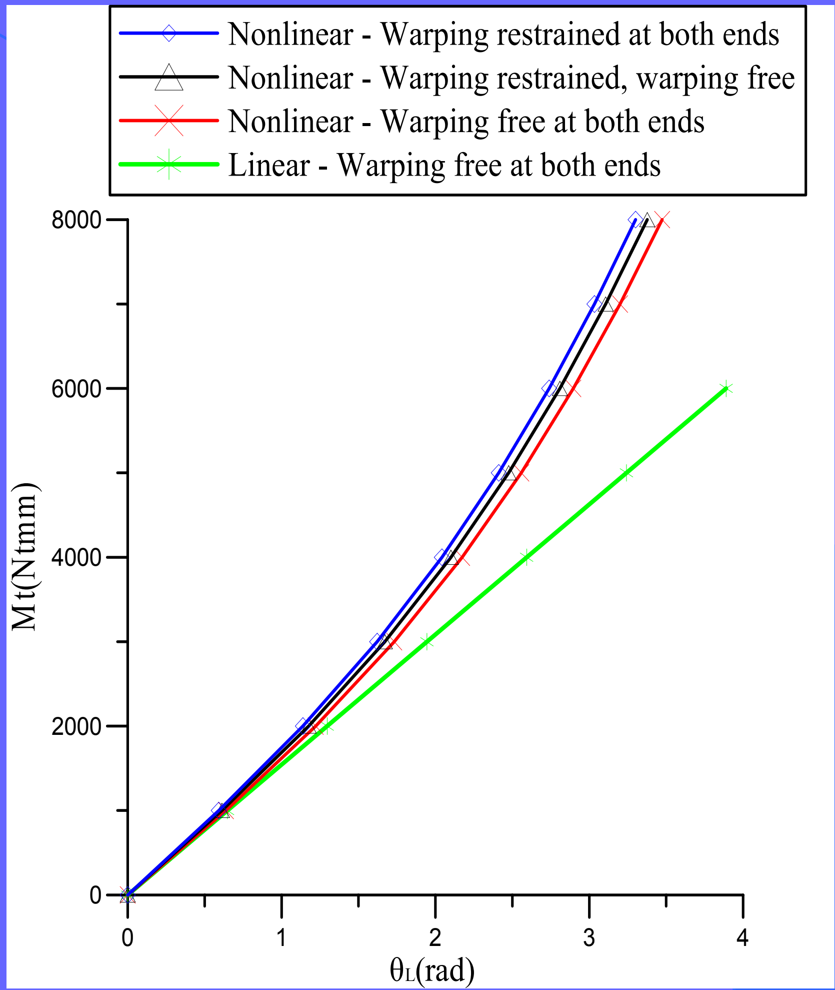
- Warping free at both ends (uniform torsion)
- Warping restrained – Warping free (cantilever)
- Warping restrained – Warping restrained

Constants	Present study	Attard (1987)
$A\left(mm^2\right)$	28,06	28,05
$I_{YY}\left(mm^4\right)$	999,2	998,0
$I_{ZZ}\left(mm^4\right)$	252,2	249,5
$I_p\left(mm^4\right)$	1991	1996
$I_{pp}\left(\times 10^5\ mm^6\right)$	2,544	2,556
$\beta_1\left(mm\right)$	-10,22	-10,33
$y_C\left(mm\right)$	5,135	5,165
$I_t\left(mm^4\right)$	8,766	8,620
$C_S\left(mm^6\right)$	152,154	-
$I_{n2}\left(\times 10^3\ mm^6\right)$	7,784	7,102

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 2 Angle section (monosymmetric)



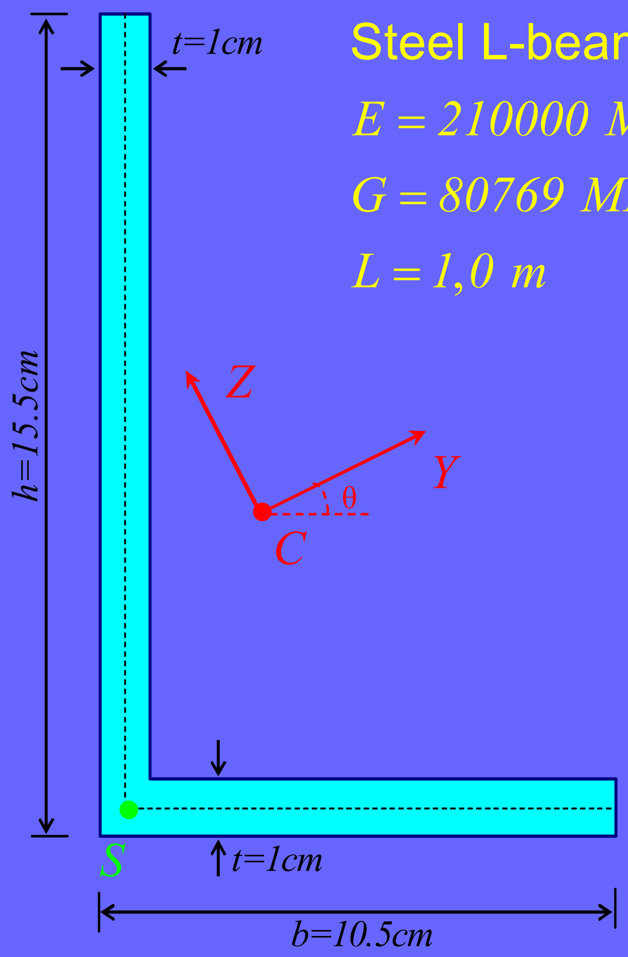


➢ Large rotations cause lateral displacements of the bar's cross sections

➢ Both geometrical nonlinearity and restraint of warping lead to stiffening of the bar

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 3   L-shaped cross-section (asymmetric)



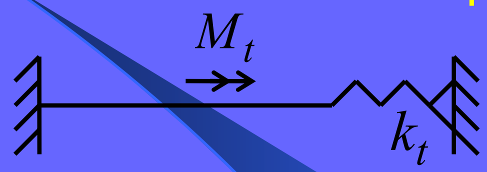
Steel L-beam:

$E = 210000\ MPa$

$G = 80769\ MPa$

$L = 1{,}0\ m$

concentrated torsional moment at the midpoint



clamped end

elastic support, warping free

$k_t = 200\ kNm/rad$

Constants of the bar			
$A(cm^2)$	25,00	$y_C(cm)$	3,688
$I_{YY}(cm^4)$	723,593	$z_C(cm)$	3,253
$I_{ZZ}(cm^4)$	132,198	$I_t(cm^4)$	8,391
$I_p(cm^4)$	1460,417	$C_S(cm^6)$	120,913
$I_{pp}(cm^6)$	172118,37	$U_w(cm^6)$	167,382
$\beta_1(cm)$	8,217	$I_n(cm^6)$	5717,765
$\beta_2(cm)$	3,950	$I_{n2}(cm^6)$	5949,475
$\theta(rad)$	0,430		

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 3   L-shaped cross-section (asymmetric)

Constants of the bar			
$A(cm^2)$	25,00	$y_C(cm)$	3,688
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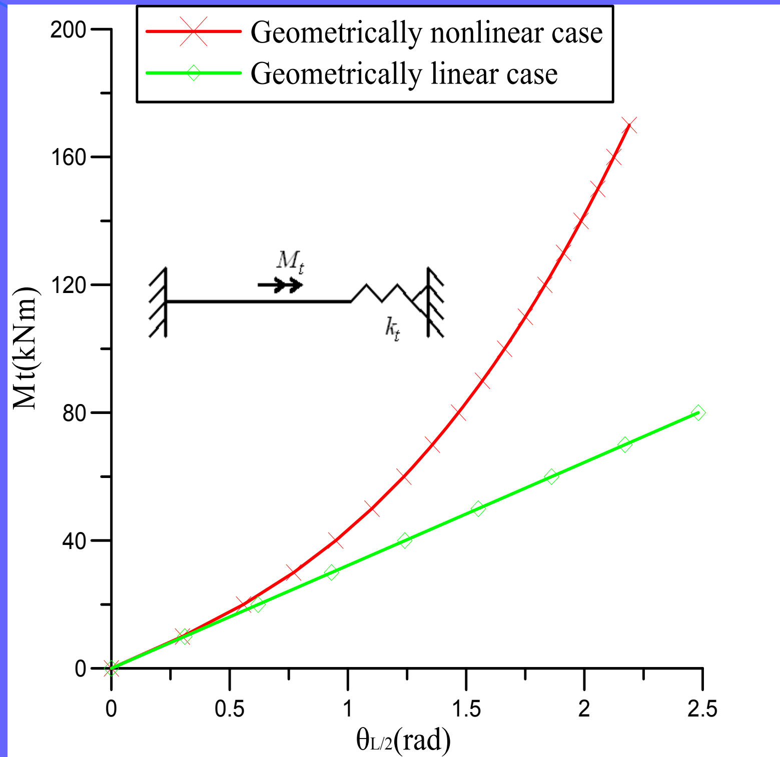
$\beta_2(cm)$	3,950	$I_{n2}(cm^6)$	5949,475
$\theta(rad)$	0,430		

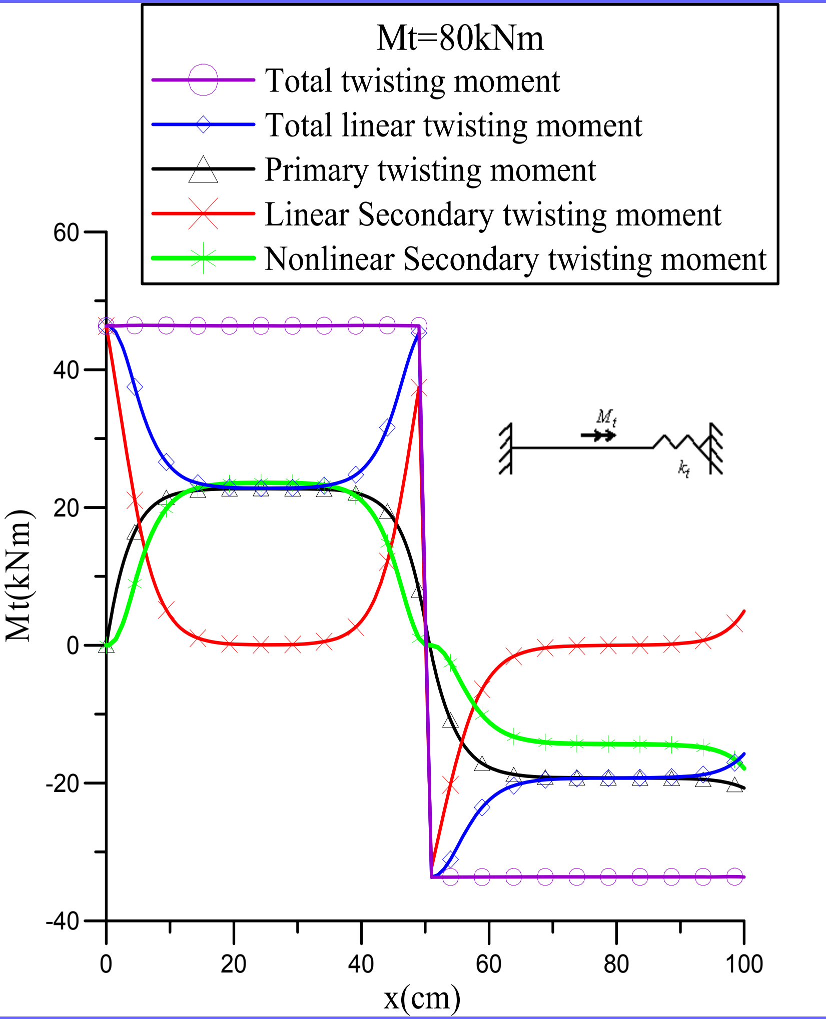
for $M_t = 170kNm$	Maximum values	Minimum values
$\theta_x\ (rad)$	2,1928	0,0000
$\theta'_x\ (rad/cm)$	0,0483	-0,0433
$\theta''_x\ (rad/cm^2)$	0,0162	-0,0130
$\theta'''_x\ (rad/cm^3)$	0,0026	-0,0041

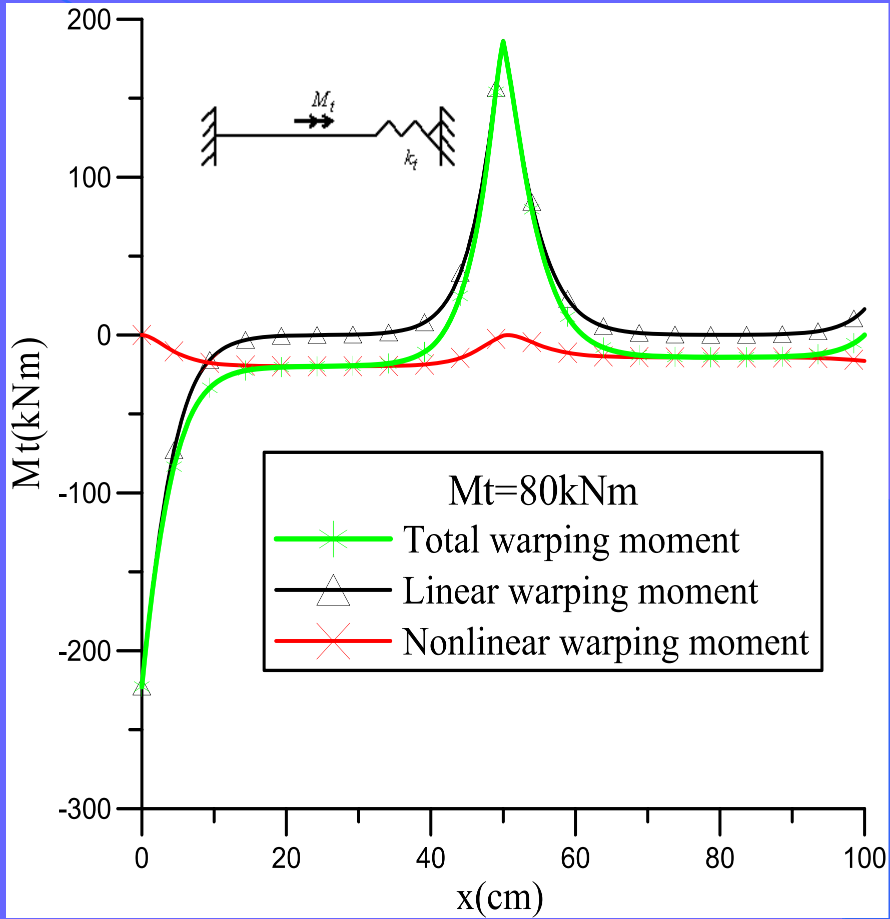


➢ Torsional rigidity is increased due to geometrical nonlinearity

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 3 L-shaped cross-section (asymmetric)



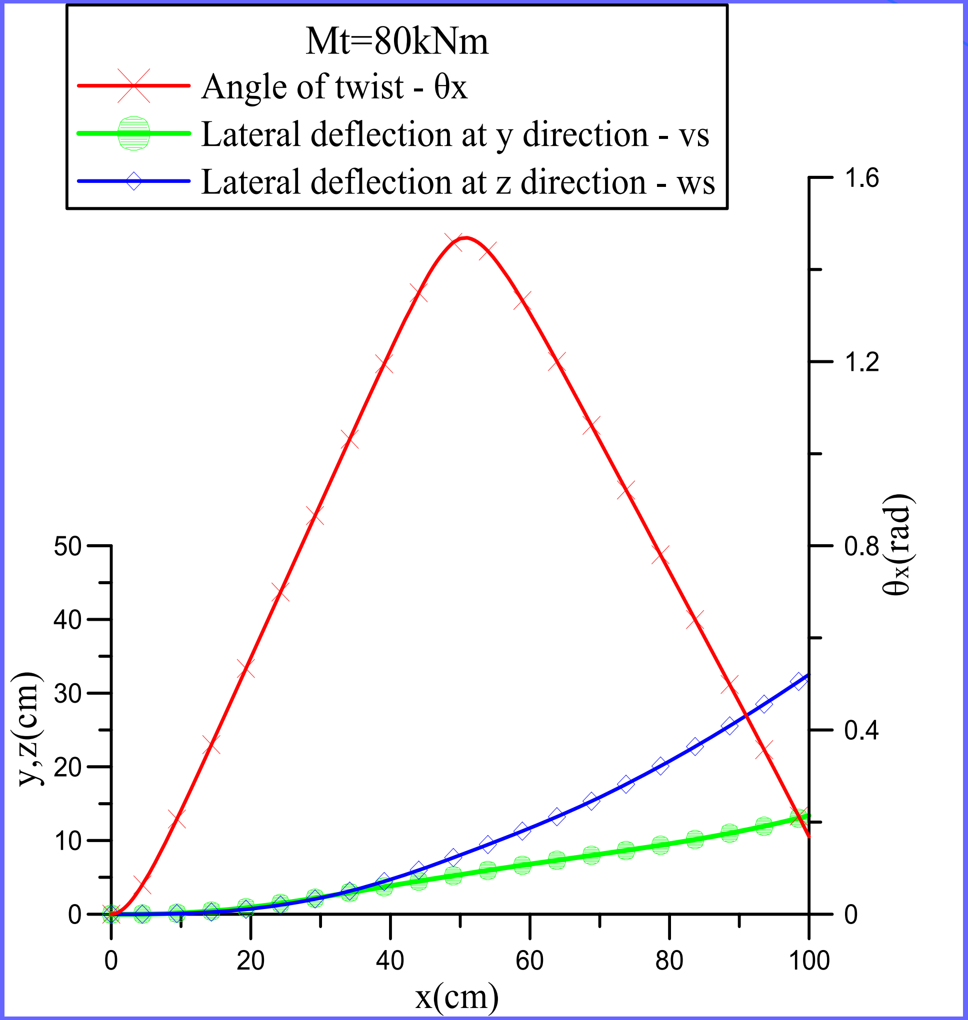


➢ The nonlinear secondary twisting moment ("Wagner" torque) reaches significant values locally along the bar

➢ The nonlinear warping moment reaches very small values

ELASTIC THEORY OF NONLINEAR NONUNIFORM TORSION

Example 3 L-shaped cross-section (asymmetric)



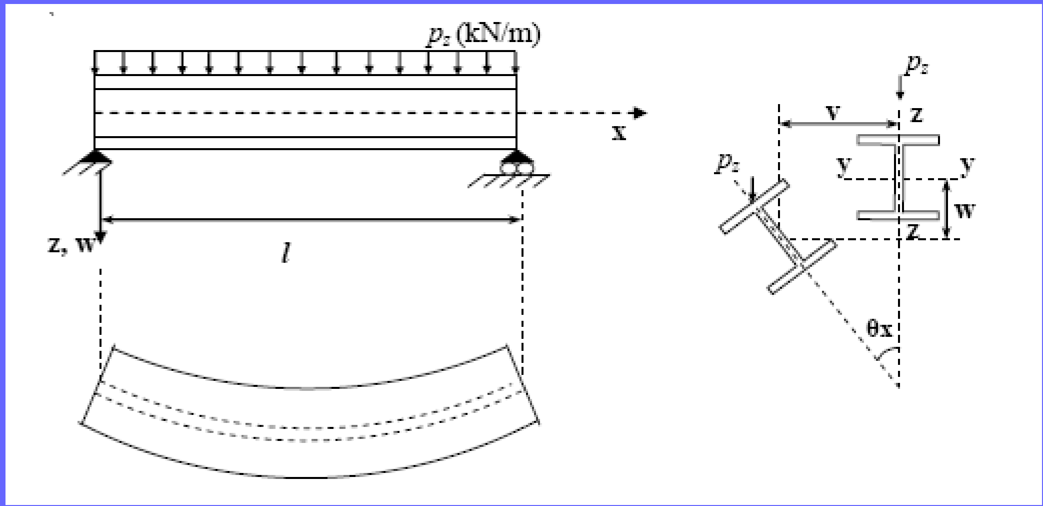
- Bars of asymmetric cross-section exhibit lateral deflections due to geometrical nonlinearity

Kinematical boundary conditions:

$$v_S(0) = w_S(0) = 0, \quad v_S'(0) = w_S'(0) = 0$$

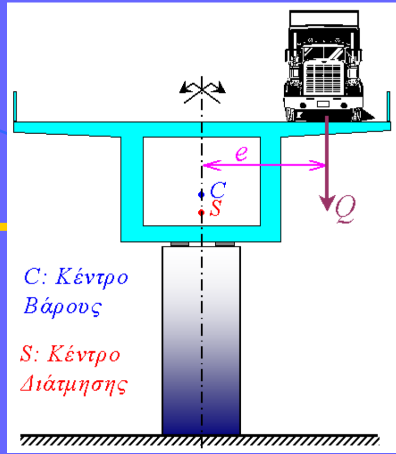
NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

- Flexural, torsional, flexural-torsional buckling of beams: In presence of a compressive axial force, the bending rigidity of the beam is reduced (equilibrium of moments in the deformed configuration). When the axial force reaches a critical value (buckling load), the beam exhibits significant flexural, torsional or flexural-torsional deformations. Postbuckling analysis investigates the equilibrium path of the buckled beam.
- Lateral-torsional buckling of beams: In presence of flexural (bending and/or transverse shear actions) loading, a beam may exhibit torsional deformations due to nonlinear coupling between flexural and torsional deformations caused by large twisting rotations. When external actions reach a critical value, the bar exhibits significant flexural-torsional (lateral-torsional) deformations. Postbuckling analysis investigates the equilibrium path of the buckled beam.

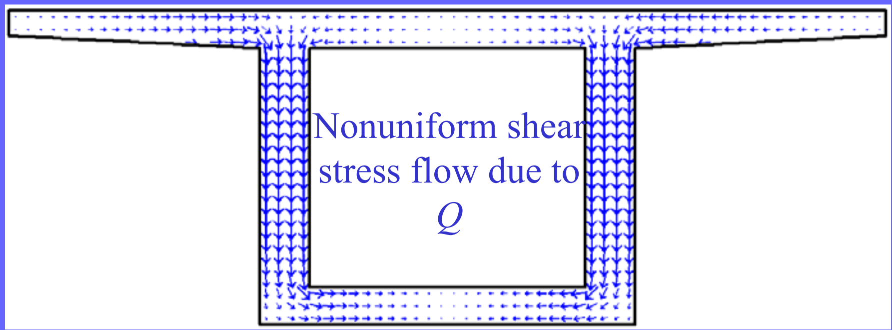


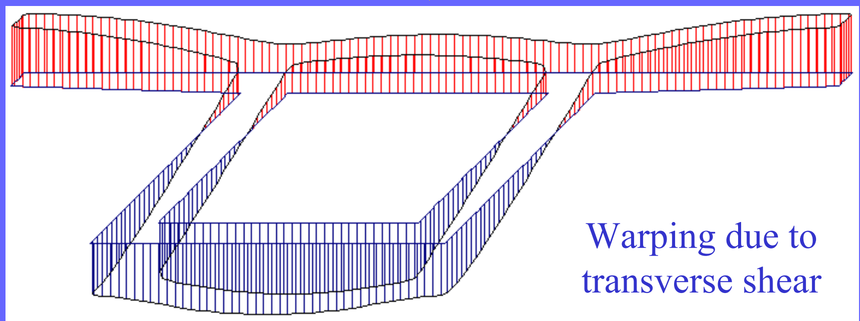
- Distortional, local buckling, etc.

Eccentric Transverse Shear Loading Q



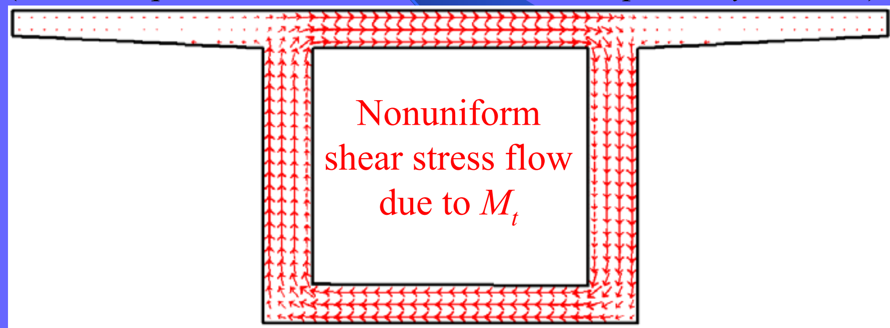
Shear Loading Q

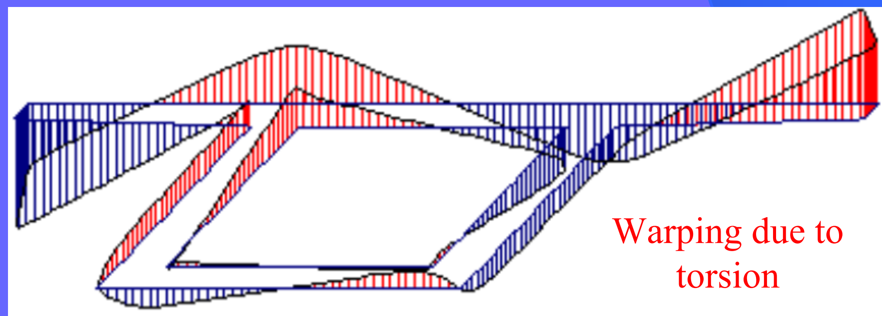




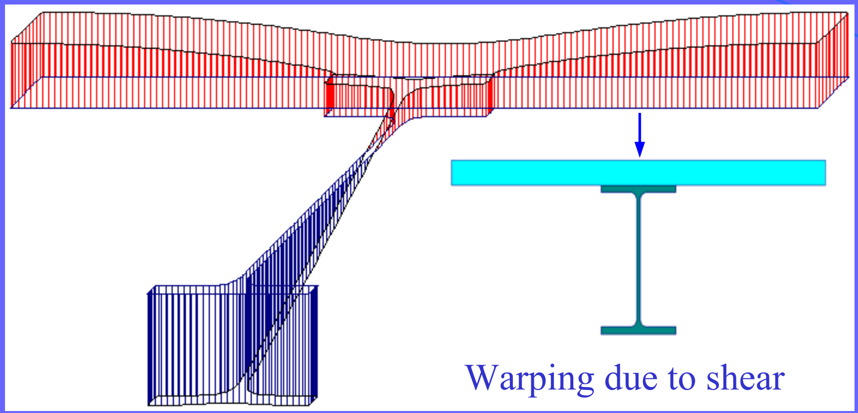
Direct Torsional Loading $M_t = Q \cdot e$)

(***Direct***: *Equilibrium Torsion*, ***Indirect***: *Compatibility Torsion*)

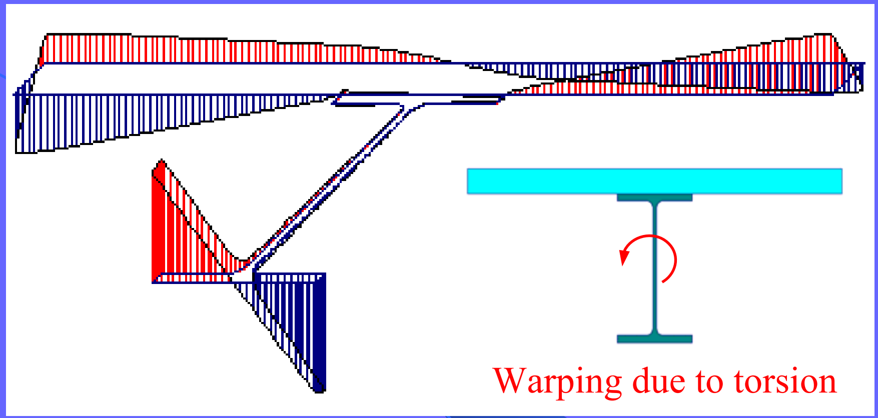




SHEAR FORCE



TWISTING MOMENT



Warping due to shear << **Warping due to torsion** *(Significant in Open Shaped Cross Sections)*

↓

- **Stress State (Stress field):**

Uniform Shear

- **Strain/Deformation State:**

Shear Deformation Coefficients *Indirect account of warping deformation (Timoshenko, 1922)*

↓

- **Stress State**

&

- **Strain State:**

Nonuniform Torsion

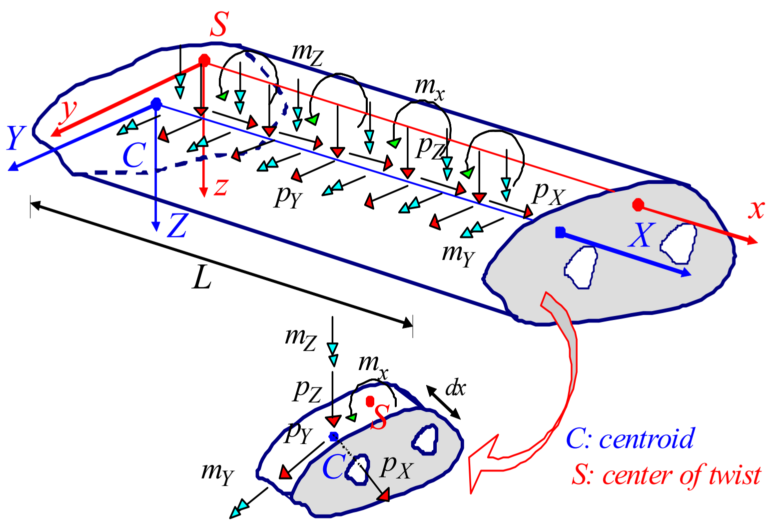
Seven degrees of freedom (14x14 [K])

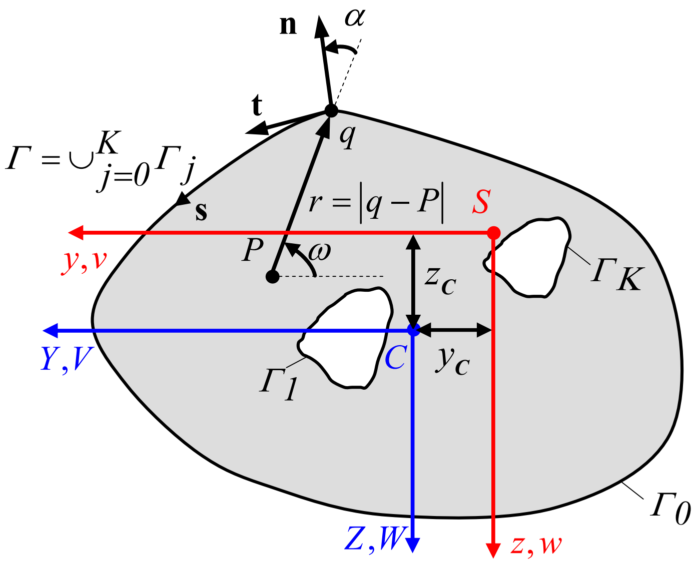
- *Additional dof.: Twisting curvature*
- *Additional stress resultant: Warping moment*

ASSUMPTIONS OF ELASTIC THEORY OF NONLINEAR BENDING INCLUDING SHEAR DEFORMATIONS

- The bar is straight.
- The bar is prismatic.
- Distortional deformations of the cross section are not allowed (cross sectional shape is not altered during deformation ($\gamma_{23}=0$, *distortion neglected*).
- Cross sections of the beam remain plane during deformation (as in Timoshenko beam theory).
- Bending rotations of the cross section and axial displacement of the beam are assumed to be small - Second order geometrically nonlinear analysis (large bending rotations are required for a postbuckling analysis).
- Twisting rotations are assumed to be small (large twisting rotations are required for a lateral-torsional analysis).
- The material of the bar is homogeneous, isotropic, continuous (no cracking) and linearly elastic: Constitutive relations of linear elasticity are valid (small strain theory).
- The distribution of stresses at the bar ends is such so that all the aforementioned assumptions are valid.

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS





Use of the principal **shear system** $CXYZ$ passing through the centroid C
(It is assumed that the center of twist S coincides with the shear center)

Displacement Field: (Arising from the plane sections hypothesis)

$$\bar{u}(x,y,z) = u(x) + \theta_Y(x)Z - \theta_Z(x)Y + \theta_x'(x)\phi_S^P(y,z)$$

$$\bar{v}(x,z) = v(x) - z\theta_x(x) \qquad \theta_Y(x) \neq -w'(x)$$

$$\bar{w}(x,y) = w(x) + y\theta_x(x) \qquad \theta_Z(x) \neq v'(x)$$

Components of the Green Strain Tensor (assumption of moderate-large deflections and small axial displacement)

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{d\theta_Y}{dx} Z - \frac{d\theta_Z}{dx} Y + \frac{1}{2}\left(\left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right)$$

$$\varepsilon_{yy} = 0 \qquad \varepsilon_{zz} = 0 \qquad \gamma_{yz} = 0$$

$$\gamma_{xy} = \frac{dv}{dx} - \theta_Z - z\frac{d\theta_x}{dx} + \frac{d\theta_x}{dx}\frac{\partial \phi_S^P}{\partial y}$$

$$\gamma_{xz} = \frac{dw}{dx} + \theta_Y + y\frac{d\theta_x}{dx} + \frac{d\theta_x}{dx}\frac{\partial \phi_S^P}{\partial z}$$

Work contributing components of the Second Piola-Kirchhoff Stress Tensor

$$S_{xx} = E\left(\frac{du}{dx} + \frac{d\theta_Y}{dx} Z - \frac{d\theta_Z}{dx} Y + \frac{d^2\theta_x}{dx^2}\left(\phi_S^P\right) + \frac{1}{2}\left(\left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 \right) \right)$$

$$S_{xy} = S_{xy}^B + S_{xy}^P = G\left(-\theta_Z + v'\right) + G\left(\theta_x{}' \cdot \left(\left(\frac{\partial \phi_S^P}{\partial y} \right) - z \right) \right)$$

$$S_{xz} = S_{xz}^B + S_{xz}^P = G\left(\theta_Y + w'\right) + G\left(\theta_x{}' \cdot \left(\left(\frac{\partial \phi_S^P}{\partial z} \right) + y \right) \right)$$

Differential Equilibrium Equations of 3D Elasticity

(Body forces neglected)

The second Piola-Kirchhoff stress components are proved that they may be introduced in the longitudinal differential equilibrium equation

Not Satisfied! → Inconsistency:

Overall equilibrium of the bar is satisfied (energy principle). The violation of the longitudinal equilibrium equation (along x) and of the associated boundary condition **is due to the unsatisfactory distribution of the shear stresses arising from the plane sections hypothesis**. Thus, in order to correct **at the global level** this unsatisfactory distribution of shear stresses, **we introduce shear correction factors in the cross sectional shear rigidities at the global equilibrium equations**

$$S_{xy} = G\left(-\theta_Z + v'\right) + G\left(\theta_x{}' \cdot \left(\left(\frac{\partial \phi_S^P}{\partial y}\right) - z\right)\right)$$

$$S_{xz} = G\left(\theta_Y + w'\right) + G\left(\theta_x{}' \cdot \left(\left(\frac{\partial \phi_S^P}{\partial z}\right) + y\right)\right)$$

Shear stresses due to torsion: self-equilibrating

$$\begin{pmatrix} \nabla^2 \phi_S^P = 0 \text{ in } \Omega \\ \dfrac{\partial \phi_S^P}{\partial n} = z \cdot n_y - y \cdot n_z \text{ on } \Gamma \end{pmatrix}$$

constant distribution: unsatisfactory

Stress Resultants

- **Shear stress resultants:** $Q_y = \int_{\Omega} S_{xy} d\Omega = \ldots = GA_y \left(\frac{dv}{dx} - \theta_Z \right)$

$$Q_z = \int_{\Omega} S_{xz} d\Omega = \ldots = GA_z \left(\theta_Y + \frac{dw}{dx} \right)$$

A_y, A_z: Shear areas with respect to the y,z axes

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad A_z = \kappa_z A = \frac{1}{a_z} A$$

κ_y, κ_z: shear correction factors (<1)

a_y, a_z: shear deformation coefficients(>1)

$\left(\textbf{From the assumed displacement field we would have obtained shear rigidities GA which are larger than the actual ones} \right)$

Since we are working with the principal shear system of axes $\Rightarrow$ $a_{yz} = 0$

Thus the relations of shear stress resultants with respect to the kinematical components are decoupled

Stress Resultants

In general we would have $a_{yz} \neq 0$:

Computation of shear deformation coefficients: Energy approach

U_{exact} → From uniform shear beam theory

U_{appr} → From this theory

$$\underbrace{\int_{\Omega} \frac{S_{xy}^2 + S_{xz}^2}{2G} d\Omega}_{U_{exact}} = \underbrace{\frac{\alpha_y Q_y{}^2}{2AG} + \frac{\alpha_z Q_z{}^2}{2AG} + \frac{\alpha_{yz} Q_y Q_z}{AG}}_{U_{appr}}$$

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad A_z = \kappa_z A = \frac{1}{a_z} A \qquad \left(A_{yz} = \kappa_{yz} A = \frac{1}{a_{yz}} A \right)$$

Stress Resultants

- **Axial force:** $N = \int_{\Omega} S_{xx} d\Omega$

The rotations of the infinitesimal surfaces comprising the cross section which occur during deformation are taken into account through the definitions of stress resultants

$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

$A = \int_{\Omega} d\Omega$: Surface area of the cross section

→ Third order geometrically nonlinear analysis (postbuckling analysis):

Requires nonlinear expressions of bending curvatures (or large bending rotations)

→ Second order geometrically nonlinear analysis (buckling analysis):

Full expression of N is required

→ Linearized second order geometrically nonlinear analysis (buckling analysis):

N is taken as $N = EA\dfrac{du}{dx}$ (principle of superposition holds)

Stress Resultants

- **Bending moments:** $M_Y = \int_{\Omega} S_{xx} Z d\Omega \qquad M_Z = -\int_{\Omega} S_{xx} Y d\Omega$

Bending moments are defined with respect to the principal shear system of axes passing through the centroid of the cross section

$$M_Y = \int_{\Omega} S_{xx} Z d\Omega = \ldots = EI_Y \frac{d\theta_Y}{dx} - EI_{YZ} \frac{d\theta_Z}{dx}$$

$$M_Z = -\int_{\Omega} S_{xx} Y d\Omega = \ldots = EI_Z \frac{d\theta_Z}{dx} - EI_{YZ} \frac{d\theta_Y}{dx}$$

$$I_Y = \int_{\Omega} Z^2 d\Omega, \; I_Z = \int_{\Omega} Y^2 d\Omega, \; I_{YZ} = \int_{\Omega} YZ d\Omega:$$

Moments of inertia with respect to the centroid of the cross section

Stress Resultants

- **Torsional stress resultants:** $M_t^P = \int_\Omega \left[S_{xy}^P \left(\frac{\partial \phi_S^P}{\partial y} - z \right) + S_{xz}^P \left(\frac{\partial \phi_S^P}{\partial z} + y \right) \right] d\Omega$

$$M_w = -\int_\Omega S_{xx} \phi_S^P \, d\Omega$$

Torsional stress resultants are defined with respect to the principal shear system of axes passing through the center of twist of the cross section

$$M_t^P = \int_\Omega \left[S_{xy}^P \left(\frac{\partial \phi_S^P}{\partial y} - z \right) + S_{xz}^P \left(\frac{\partial \phi_S^P}{\partial z} + y \right) \right] d\Omega = \ldots = GI_t \frac{d\theta_x}{dx}$$

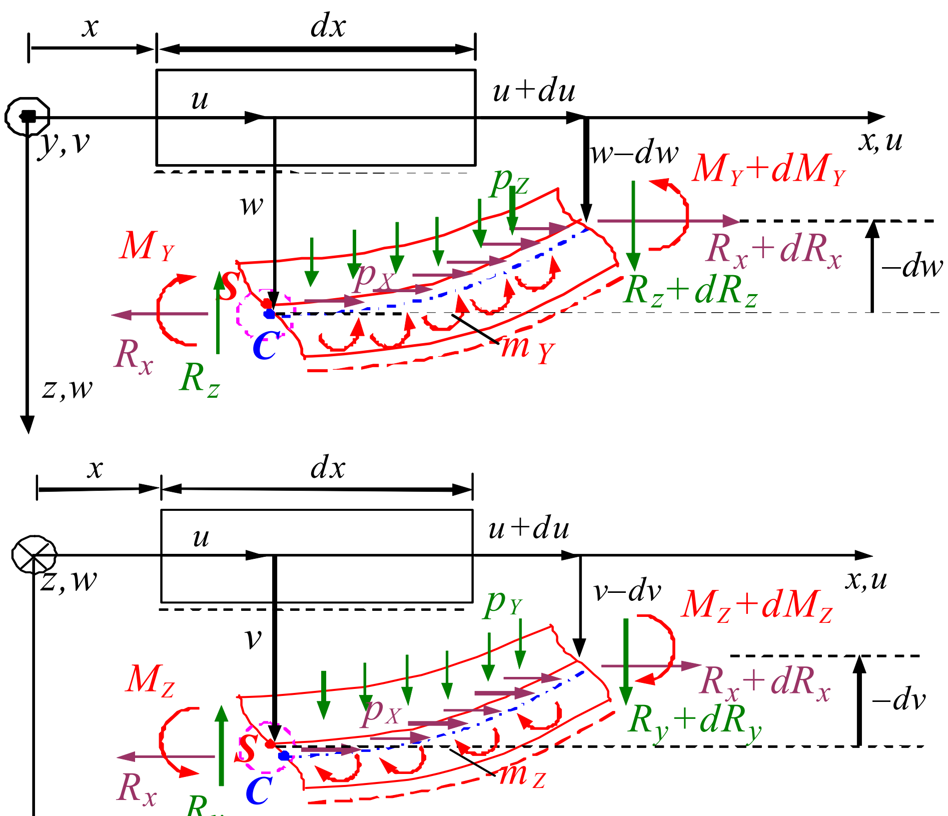
$$M_w = -\int_\Omega S_{xx} \phi_S^P \, d\Omega = \ldots = -EC_S \frac{d^2\theta_x}{dx^2}$$

$$I_t = \int_\Omega \left(y^2 + z^2 + y \frac{\partial \phi_S^P}{\partial z} - z \frac{\partial \phi_S^P}{\partial y} \right) d\Omega, \; C_S = \int_\Omega \left(\phi_S^P \right)^2 d\Omega:$$

Primary torsion constant and warping constant with respect to the center of twist of the cross section

Global Equilibrium Equations & Boundary conditions

Method of Equilibrium ↓ or Energy Method ↓



TOTAL POTENTIAL ENERGY

$$\frac{\partial F}{\partial \theta_1} - \frac{d}{d\tilde{x}_1}\frac{\partial F}{\partial \theta_1'} + \frac{d^2}{d\tilde{x}_1^{\,2}}\frac{\partial F}{\partial \theta_1''} = 0$$

(*Euler–Lagrange eqns*)

Equilibrium of axial forces

$$\frac{dR_X}{dx} + p_X = 0$$

$$R_X = N + Q_z\varphi_Y + Q_y\varphi_Z \approx N$$

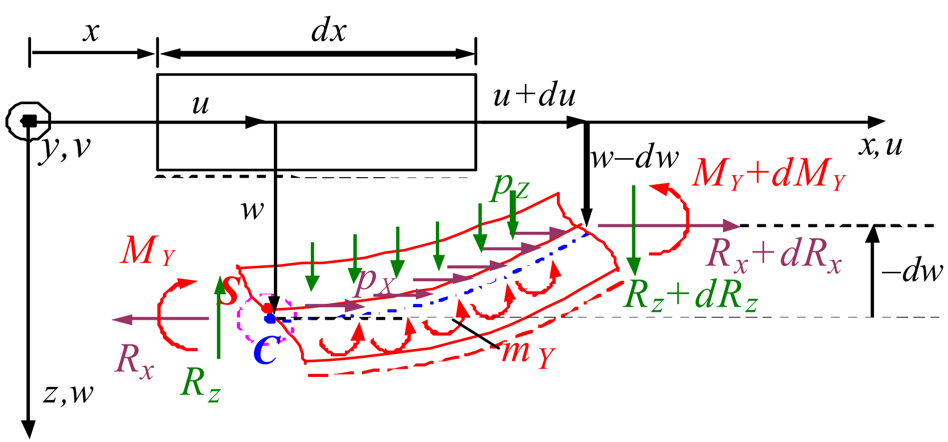
$$\frac{dN}{dx} + p_X = 0$$

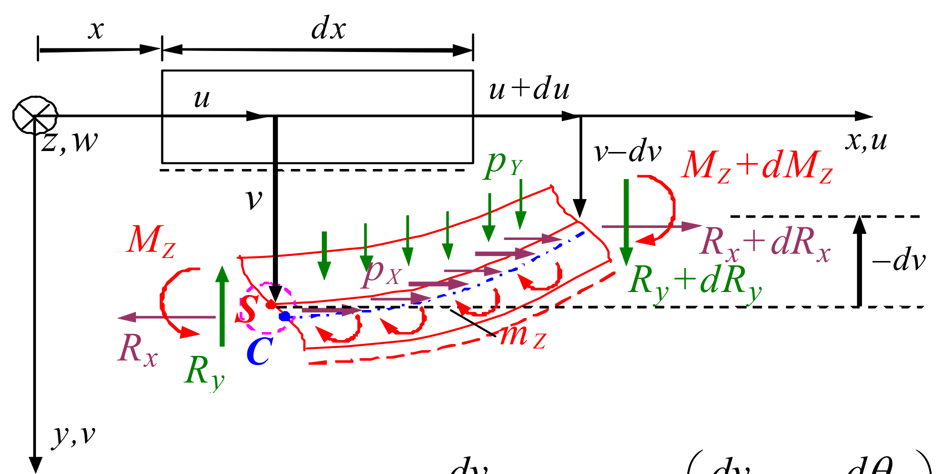
→ If the beam's axial force can be determined solely from equilibrium requirements then the second order and linearized second order theories yield the same deflections (only axial shortening of the beam is different)

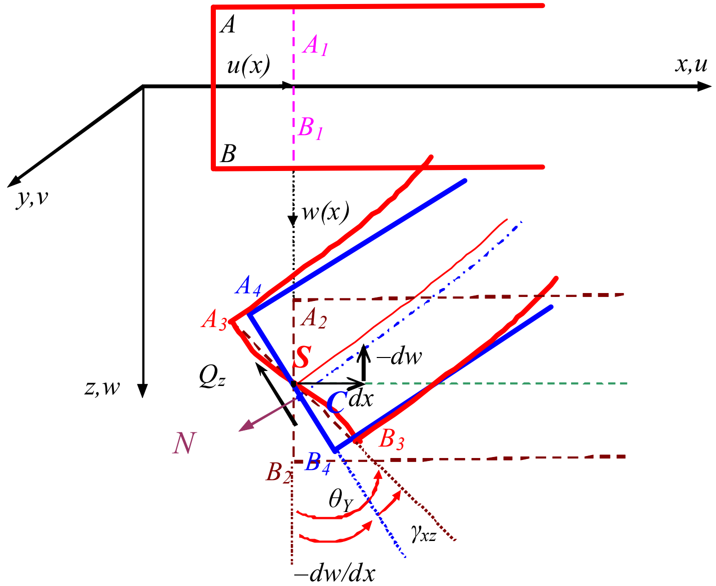
NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Global Equilibrium Equations & Boundary conditions

Method of Equilibrium







Equilibrium of bending moments

$$\frac{dM_Y}{dx} - Q_z + m_Y = 0 \qquad \frac{dM_Z}{dx} + Q_y + m_Z = 0$$

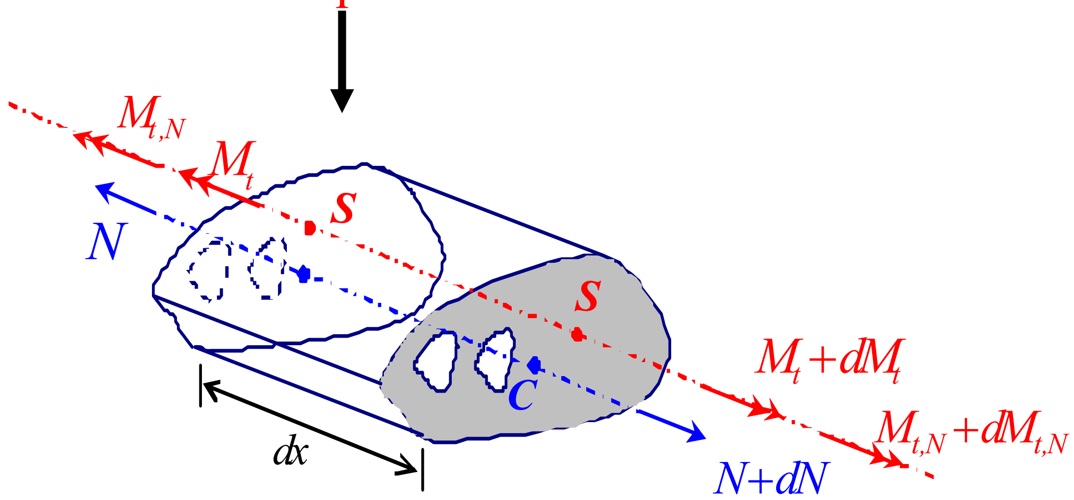
Equilibrium of transverse shear forces

$$\frac{dR_y}{dx} + p_Y = 0 \qquad \frac{dR_z}{dx} + p_Z = 0$$

$$R_y = Q_y + N\frac{dv_C}{dx} = Q_y + N\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) \qquad R_z = Q_z + N\frac{dw_C}{dx} = Q_z + N\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right)$$

Global Equilibrium Equations & Boundary conditions

Method of Equilibrium



Equilibrium of torsional moments

$$-\left(\frac{dM_t}{dx}+\frac{dM_{t,N}}{dx}\right)=-\left(\frac{dM_t^S}{dx}+\frac{dM_t^P}{dx}+\frac{dM_{t,N}}{dx}\right)=m_x+p_Z y_C-p_Y z_C \qquad \frac{dM_w}{dx}=M_t^S$$

$$-\left(\frac{dM_t}{dx}+\frac{dM_{t,N}}{dx}\right)=-\left(\frac{d^2M_w}{dx^2}+\frac{dM_t^P}{dx}+\frac{dM_{t,N}}{dx}\right)=m_x+p_Z y_C-p_Y z_C$$

Global Equilibrium Equations & Boundary conditions

Equilibrium of axial forces

$$\frac{dN}{dx} = -p_X \Rightarrow EA\left[\frac{d^2u}{dx^2} + \frac{d^2w}{dx^2}\frac{dw}{dx} + \frac{d^2v}{dx^2}\frac{dv}{dx}\right] = -p_x \quad (1)$$

Equilibrium of torsional moments

$$-\left(\frac{dM_t}{dx} + \frac{dM_{t,N}}{dx}\right) = -\left(\frac{d^2M_w}{dx^2} + \frac{dM_t^P}{dx} + \frac{dM_{t,N}}{dx}\right) = m_x + p_Z y_C - p_Y z_C \Rightarrow$$

$$EC_S\frac{d^4\theta_x}{dx^4} - GI_t\frac{d^2\theta_x}{dx^2} - N\left(y_C\frac{d^2w}{dx^2} - z_C\frac{d^2v}{dx^2} + \frac{I_S}{A}\frac{d^2\theta_x}{dx^2}\right) =$$

Inside the bar interval

$$m_x + p_Z y_C - p_Y z_C - p_X\left(y_C\frac{dw}{dx} - z_C\frac{dv}{dx} + \frac{I_S}{A}\frac{d\theta_x}{dx}\right) \quad (2)$$

$$a_1 u + a_2 N = a_3$$

$$\delta_1\theta_x + \delta_2 M_t = \delta_3 \qquad \bar{\delta}_1\frac{d\theta_x}{dx} + \bar{\delta}_2 M_w = \bar{\delta}_3$$

At the bar ends

→ Coupled system of equations due to geometrical nonlinearity

Global Equilibrium Equations & Boundary conditions

Equilibrium of bending moments

$$\frac{dM_Y}{dx} - Q_z + m_Y = 0 \Rightarrow EI_Y \frac{d^2\theta_Y}{dx^2} - EI_{YZ}\frac{d^2\theta_Z}{dx^2} - \frac{GA}{a_Z}\left(\theta_Y + \frac{dw}{dx}\right) + m_Y = 0 \quad (3)$$

$$\frac{dM_Z}{dx} + Q_y + m_Z = 0 \Rightarrow EI_Z \frac{d^2\theta_Z}{dx^2} - EI_{YZ}\frac{d^2\theta_Y}{dx^2} + \frac{GA}{a_Y}\left(\frac{dv}{dx} - \theta_Z\right) + m_Z = 0 \quad (4)$$

Equilibrium of transverse shear forces

Inside the bar interval

$$\frac{dR_y}{dx} + p_Y = 0 \Rightarrow \frac{GA}{a_Y}\left(\frac{d^2v}{dx^2} - \frac{d\theta_Z}{dx}\right) + \frac{dN}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + p_Y = 0 \quad (5)$$

$$\frac{dR_z}{dx} + p_Z = 0 \Rightarrow \frac{GA}{a_Z}\left(\frac{d\theta_Y}{dx} + \frac{d^2w}{dx^2}\right) + \frac{dN}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + p_Z = 0 \quad (6)$$

$$\beta_1 v + \beta_2 R_y = \beta_3 \qquad \gamma_1 w + \gamma_2 R_z = \gamma_3$$

$$\bar{\beta}_1\theta_Z + \bar{\beta}_2 M_Z = \bar{\beta}_3 \qquad \bar{\gamma}_1\theta_Y + \bar{\gamma}_2 M_Y = \bar{\gamma}_3$$

At the bar ends

$\rightarrow$ Coupled system of equations due to principal shear system of axes, shear deformation effects and geometrical nonlinearity

Global Equilibrium Equations & Boundary conditions

Combination of equations may be performed in order to uncouple the problem unknowns - Solution with respect to deflections

Resolution of deflections, twisting rotation and axial displacement:

$$EA\left[\frac{d^2u}{dx^2}+\frac{d^2w}{dx^2}\frac{dw}{dx}+\frac{d^2v}{dx^2}\frac{dv}{dx}\right]=-p_x$$

$$EI_{ZZ}\frac{d^4v}{dx^4}+EI_{YZ}\frac{d^4w}{dx^4}+\alpha_y\frac{EI_{ZZ}}{GA}\left[\frac{d^2p_Y}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)-3p_X\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)\right]+$$
$$+\alpha_z\frac{EI_{YZ}}{GA}\left[\frac{d^2p_Z}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)-3p_X\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)\right]-p_Y+p_X\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-$$
$$-N\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)=0$$

Inside the bar interval

$$EI_{YY}\frac{d^4w}{dx^4}+EI_{YZ}\frac{d^4v}{dx^4}+\alpha_z\frac{EI_{YY}}{GA}\left[\frac{d^2p_Z}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)-3p_X\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)\right]+$$
$$+\alpha_y\frac{EI_{YZ}}{GA}\left[\frac{d^2p_Y}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)-3p_X\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)\right]-p_Z+p_X\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-$$
$$-N\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)=0$$

$$EC_S\frac{d^4\theta_x}{dx^4}-GI_t\frac{d^2\theta_x}{dx^2}-N\left(y_c\frac{d^2w}{dx^2}-z_c\frac{d^2v}{dx^2}+\frac{I_S}{A}\frac{d^2\theta_x}{dx^2}\right)=m_x+p_Zy_C-p_Yz_C-p_X\left(y_c\frac{dw}{dx}-z_c\frac{dv}{dx}\right)-p_X\frac{I_S}{A}\frac{d\theta_x}{dx}$$

Global Equilibrium Equations & Boundary conditions

Combination of equations may be performed in order to uncouple the problem unknowns - Solution with respect to deflections

Resolution of deflections, twisting rotations and axial displacement:

$$a_1 u + a_2 N = a_3 \qquad \beta_1 v + \beta_2 R_y = \beta_3 \qquad \bar{\beta}_1 \theta_Z + \bar{\beta}_2 M_Z = \bar{\beta}_3 \qquad \gamma_1 w + \gamma_2 R_z = \gamma_3$$

$$\bar{\gamma}_1 \theta_Y + \bar{\gamma}_2 M_Y = \bar{\gamma}_3 \qquad \delta_1 \theta_x + \delta_2 M_t = \delta_3 \qquad \bar{\delta}_1 \frac{d\theta_x}{dx} + \bar{\delta}_2 M_w = \bar{\delta}_3$$

At the bar ends

where:

$$\theta_Y = -\frac{dw}{dx} - \alpha_Z \frac{EI_Y}{GA}\frac{d^3w}{dx^3} - \alpha_Z \frac{EI_{YZ}}{GA}\frac{d^3v}{dx^3} + a_Z \frac{m_Y}{GA} -$$

$$-\alpha_Z^2 \frac{EI_Y}{G^2A^2}\left[\frac{dp_Z}{dx} - \frac{dp_X}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right)\right] -$$

$$-\alpha_Y\alpha_Z \frac{EI_{YZ}}{G^2A^2}\left[\frac{dp_Y}{dx} - \frac{dp_X}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right)\right]$$

$$\theta_Z = \frac{dv}{dx} + \alpha_Y \frac{EI_Z}{GA}\frac{d^3v}{dx^3} + \alpha_Y \frac{EI_{YZ}}{GA}\frac{d^3w}{dx^3} + a_Y \frac{m_Z}{GA} +$$

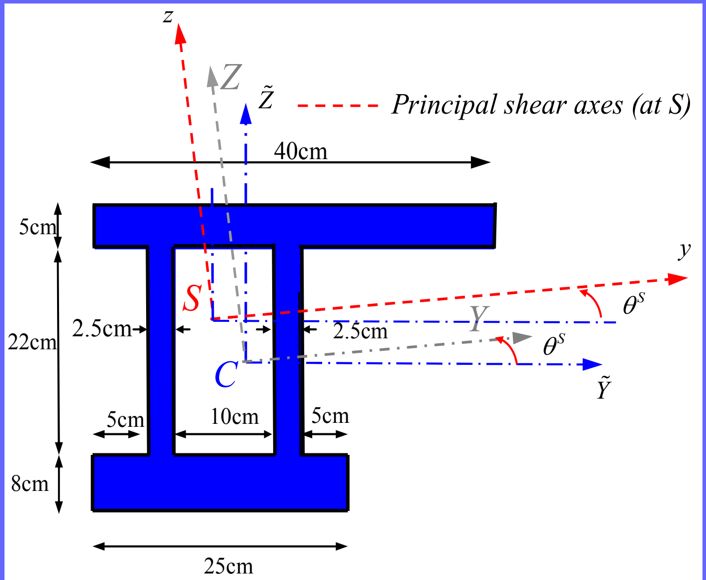
$$+\alpha_Y^2 \frac{EI_Z}{G^2A^2}\left[\frac{dp_Y}{dx} - \frac{dp_X}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right)\right] +$$

$$+\alpha_Z\alpha_Y \frac{EI_{YZ}}{G^2A^2}\left[\frac{dp_Z}{dx} - \frac{dp_X}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right)\right]$$

Resolution of bending rotations: With the above expressions

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 1 Simply supported asymmetric cross section beam



$E = 3.0 \times 10^7 \; kN / m^2$

$\nu = 0.20$

$A = 0.051 m^2$

$C_s = 4.6961 \times 10^{-6} \; m^6$

$I_t = 2.6925 \times 10^{-4} \; m^4$

$I_S = 1.58159 \times 10^{-3} \; m^4$

$l = 1.00 m$

It is worth noting that all the geometric constants and shear deformation coefficients of the cross section should be evaluated with respect to the principal shear coordinate system which does not coincide with the principal bending one

$$\tan 2\theta^S = \frac{2 a_{\tilde{y}\tilde{z}}}{a_{\tilde{y}} - a_{\tilde{z}}}$$

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 1 Simply supported asymmetric cross section beam

In the case of simply supported beam the analytical solution can be obtained by setting in the differential equations the following expressions of displacement

$$v = A_1 \sin\frac{\pi x}{l} \qquad w = A_2 \sin\frac{\pi x}{l} \qquad \theta_x = A_3 \sin\frac{\pi x}{l}$$

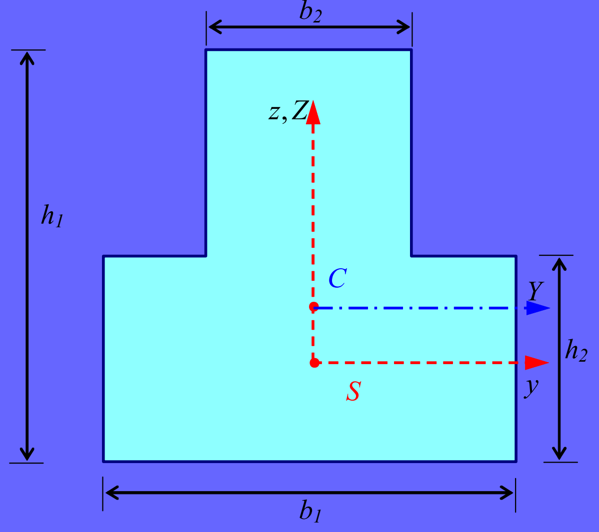
$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

system of three equations where K_{ij} are functions of P

	Without Shear Deformation		With Shear Deformation		reduction
	analytical	computed	analytical	computed	
P_1	113073	113077	81878	81882	28%
P_2	139597	139602	100607	100610	
P_3	374194	374206	228033	228040	

The effect of shear deformation is critical for the stability of the beam. The actual compressive load that causes the buckling of the beam (P_1) is smaller than the load we calculate when the shear deformation is ignored.

Example 2 Monosymmetric cross section beam



$E = 3.0 \times 10^7\, kN/m^2$

$G = 1.25 \times 10^7\, kN/m^2$

$b_1 = h_1 = 0.4m$

$l = 1.0m$

Three cases examined:

$b_2 = h_2 = 2\ cm$

$b_2 = h_2 = 8\ cm$

$b_2 = h_2 = 20\ cm$

Various boundary conditions

- **hinged – hinged**
- **fixed – hinged**
- **fixed – fixed**

The results have been compared with the corresponding values of buckling load arising from the Thin Tube Theory for the case where the shear deformation is neglected.

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 2 Monosymmetric cross section beam

Boundary Conditions *hinged-hinged*	without shear deformation		with shear deformation
	TTT	Present study	Present study
$b_2=h_2=2$ cm	970	975	972
$b_2=h_2=8$ cm	53774	54960	51169
$b_2=h_2=20$ cm	327887	350017	266234

Boundary Conditions *fixed-hinged*	without shear deformation		With shear deformation
	TTT	Present study	Present study
$b_2=h_2=2$ cm	1134	1139	1133
$b_2=h_2=8$ cm	67177	67352	62134
$b_2=h_2=20$ cm	603002	679002	394392

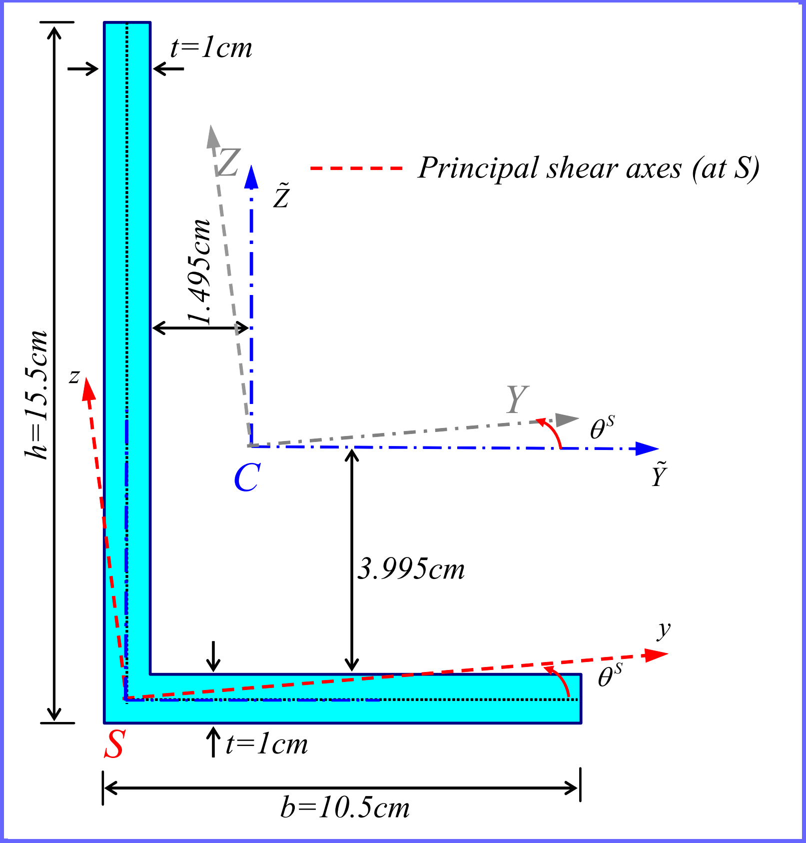
Boundary Conditions *fixed-fixed*	without shear deformation		with shear deformation
	TTT	Present study	Present study
$b_2=h_2=2$ cm	1436	1439	1432
$b_2=h_2=8$ cm	86145	84874	78782
$b_2=h_2=20$ cm	962711	919375	506198

In the third case the cross section is no longer thin walled. As a result, the thin tube theory cannot give accurate results.

The ignorance of shear deformation can lead to incorrect results which can be critical for the stability of the structure.

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 3 Asymmetric cross section beam



length: 1.00 m $E=2.1x10^8$ kN/m^2

v=0.3   Various Boundary Conditions

1st Case: ***t = 1 cm***

2nd Case: ***t = 4 cm***

It is worth noting that all the geometric constants and shear deformation coefficients of the cross section should be evaluated with respect to the principal shear coordinate system which does not coincide with the principal bending one

$$tan\, 2\theta^S = \frac{2a_{\tilde{y}\tilde{z}}}{a_{\tilde{y}} - a_{\tilde{z}}}$$

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 3 Asymmetric cross section beam

Buckling Load (kN)

t	Hinged-Hinged		Fixed-Hinged		Fixed-Fixed	
	without shear deformation	with shear deformation	without shear deformation	with shear deformation	without shear deformation	with shear deformation
1 cm	1096	1086	1237	1225	1403	1391
reduction	0.9 %		1 %		1 %	
4 cm	9374	9124	18644	17590	34341	31295
reduction	3 %		6%		9%	

The influence of the shear deformation effect on the buckling load increased due to the change of thickness of the cross section from 1cm to 4 cm. Actually, this is a result of the increase of stiffness of the beam and therefore, shear deformation becomes more critical than bending.

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 4 Clamped Beam with Doubly Symmetric Cross Section

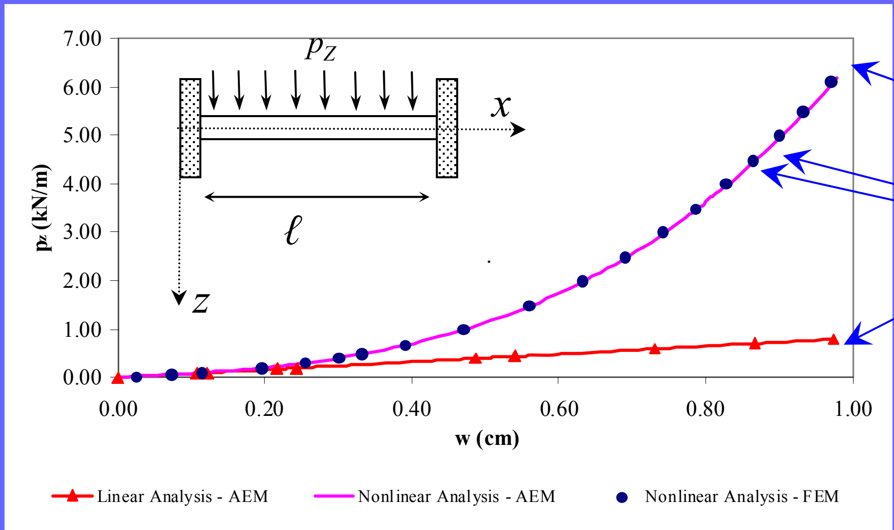
Elasticity Modulus $E = 207GPA$ $\nu = 0.3$

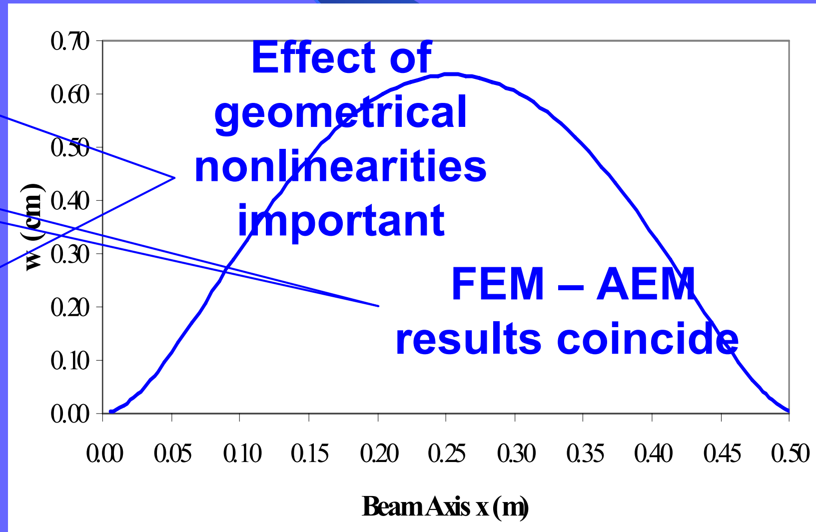
Beam Length $l = 0.508m$ Rectangular Cross Section $b_y \times h_z = 25.4mm \times 3.175mm$

Uniformly Distributed Load p_Z

(displacement at middle point)

Induced axial load at the bar (due to clamped edges)





for p_z=2.0kN/m *max w=6.31mm* (FEM) *max w=6.36mm* (AEM)

Axial Force $N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] = \frac{EA}{2l}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx$

Shear Deformation Effect can be ignored

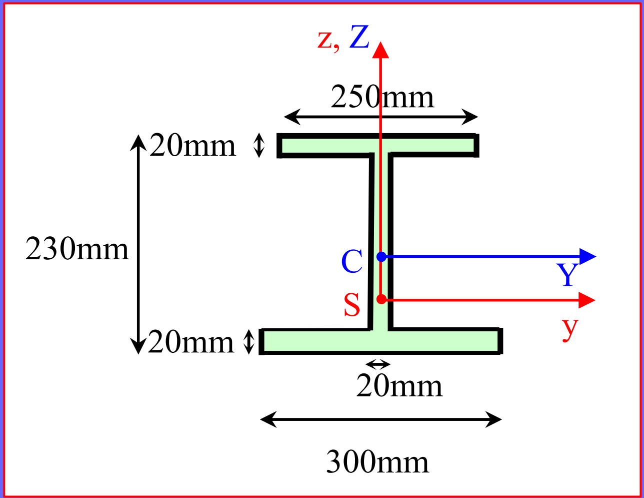
without shear	*with shear*
max w=9.76mm	*max w=9.77mm*

Shear Def. Coefficients $\alpha_Y = 1.20$ $\alpha_Z = 3.87$

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 5 Clamped Beam with Monosymmetric Cross Section

Beam Length $l = 4.5m$ **Shear Def. Coefficients** $a_Y = 1.63 \quad a_Z = 3.93$



Loading

Uniformly distributed transverse loading $p_Z = p_Y$ applied at the cross section's centroid

$$A = 1.48 \times 10^{-2}\, m^2 ,\ I_Y = 1.323 \times 10^{-4}\, m^4 ,\ I_Z = 7.117 \times 10^{-5}\, m^4 ,\ C_s = 7.22 \times 10^{-7}\, m^6 ,$$

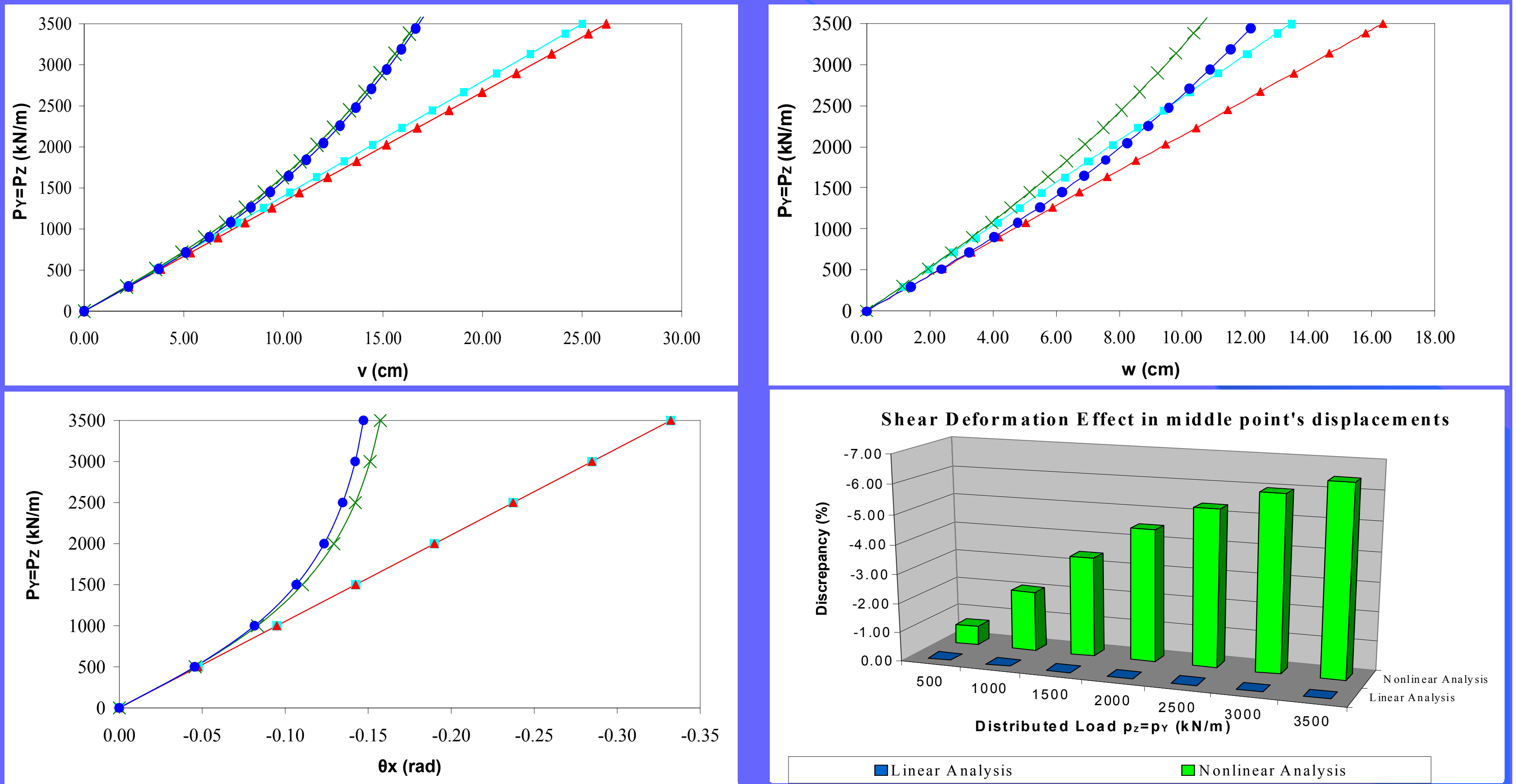
$$I_t = 2.00 \times 10^{-6}\, m^4 ,\ a_Y = 1.63 ,\ a_Z = 3.93 ,\ z_C = 2.07 \times 10^{-2}\, m ,$$

$$E = 2.1 \times 10^8\, kN / m^2 ,\ v = 0.3$$

$$N = EA\left[u' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] = \frac{EA}{2l}\int_0^l \left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right) dx$$

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 5 Clamped Beam with Monosymmetric Cross Section

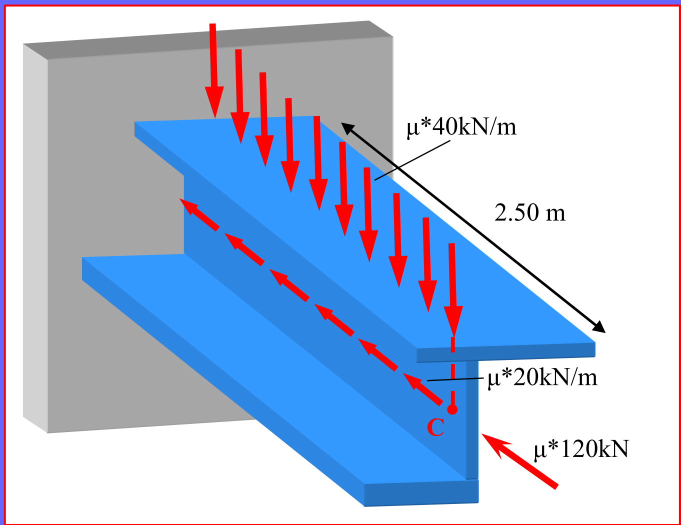


(displacements at the middle point of the beam)

Linear Analysis-AEM-Without shear def.
Linear Analysis-AEM-With shear def.
NonLinear Analysis-AEM-Without shear def.
NonLinear Analysis-AEM-With shear def.

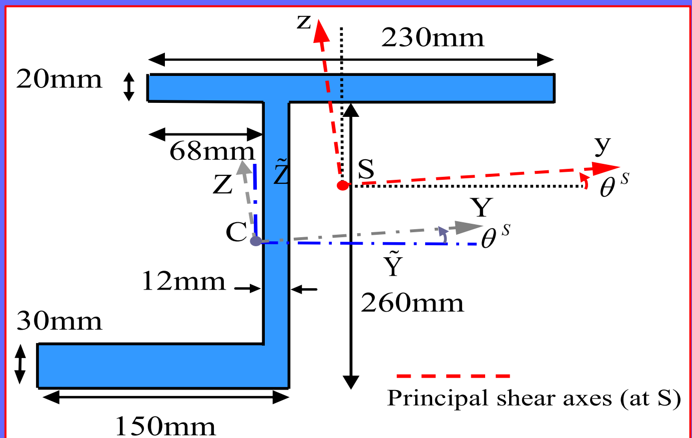
Example 6 Cantilever with asymmetric cross section subjected to distributed transverse and axial loading

Scale factor μ



Basic load:

- distributed transverse load $p_{\tilde{Z}} = -40kN/m$
- axial load $p_X = -20kN/m$
- concentrated axial load $P_X = -120kN$

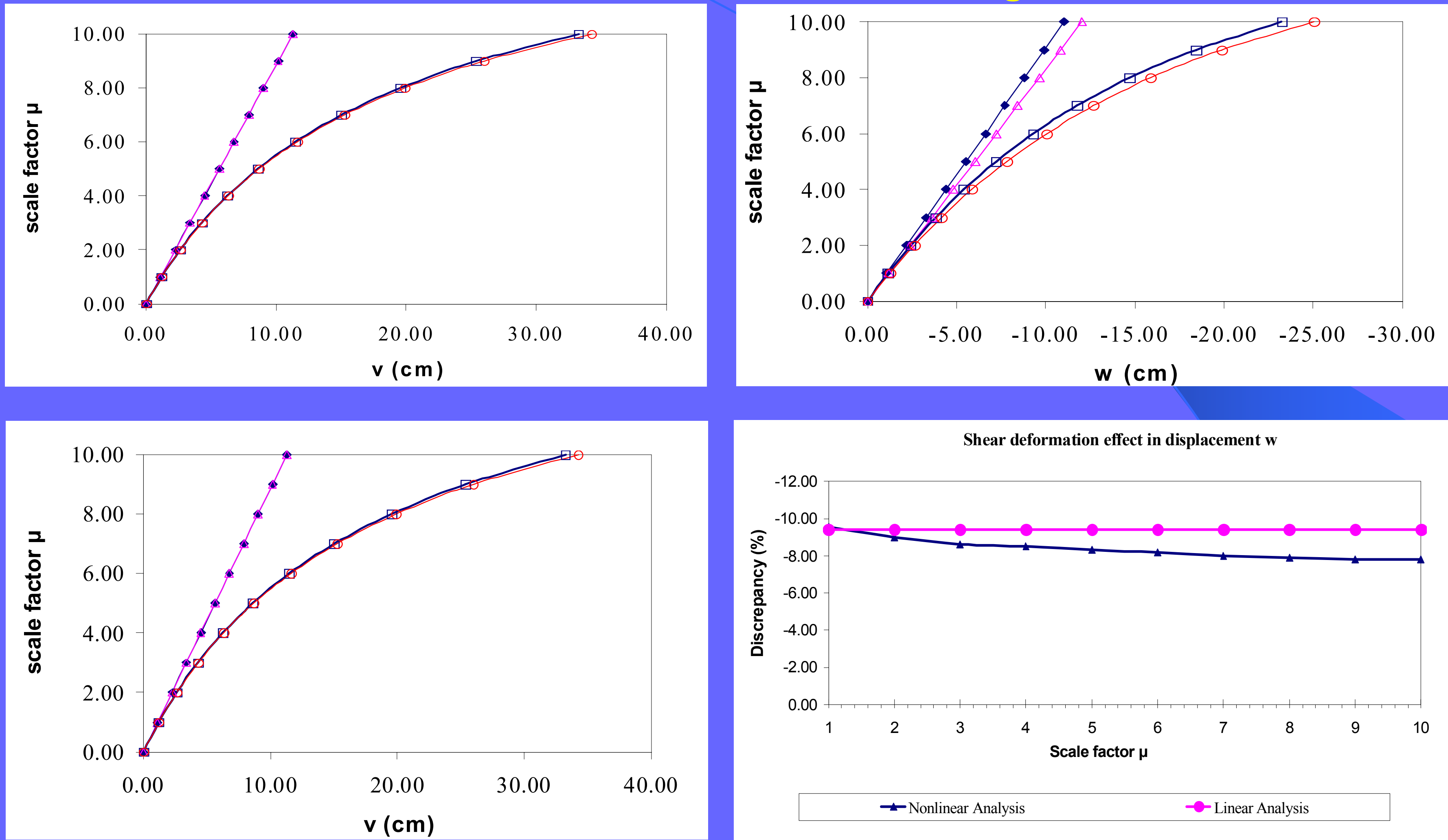


Geometric constants and shear deformation coefficients of the examined cross section

Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 1.606\times10^{-4}\,m^4$	$I_{YY} = 1.545\times10^{-4}\,m^4$
$I_{\tilde{Z}\tilde{Z}} = 5.665\times10^{-5}\,m^4$	$I_{ZZ} = 6.278\times10^{-5}\,m^4$
$I_{\tilde{Y}\tilde{Z}} = 6.384\times10^{-5}\,m^4$	$I_{YZ} = 6.837\times10^{-5}\,m^4$
$\alpha_{\tilde{y}} = 1.741$	$\alpha_y = 1.736$
$\alpha_{\tilde{z}} = 3.902$	$\alpha_z = 3.907$
$\alpha_{\tilde{y}\tilde{z}} = -0.10$	$\alpha_{yz} = 0.0$
$\tilde{y}_C = -3.84\times10^{-2}\,m$	$y_C = -4.01\times10^{-2}\,m$
$\tilde{z}_C = -3.79\times10^{-2}\,m$	$z_C = -3.61\times10^{-2}\,m$
$\theta^S = 0.046\,rad$	-

NONLINEAR BENDING OF ELASTIC BEAMS INCLUDING SHEAR DEFORMATIONS

Example 6 Cantilever with asymmetric cross section subjected to distributed transverse and axial loading



(displacements at the middle point of the beam)

- Linear Analysis-AEM-without shear def.
- Linear Analysis-AEM-with shear def.
- Nonlineas Analysis-AEM without shear def.
- Nonlineas Analysis-AEM-with shear def.

THANK YOU

FOR YOUR ATTENTION