NATIONAL TECHNICAL UNIVERSITY OF ATHENS

SCHOOL OF CIVIL ENGINEERING

INTER-DEPARTMENTAL POSTGRADUATE COURSES PROGRAMMES

«ΔΟΜΟΣΤΑΤΙΚΟΣ ΣΧΕΔΙΑΣΜΟΣ ΚΑΙ ΑΝΑΛΥΣΗ ΚΑΤΑΣΚΕΥΩΝ»

"ANALYSIS AND DESIGN OF EARTHQUAKE RESISTANT STRUCTURES"

NONUNIFORM TORSION, UNIFORM SHEAR AND TIMOSHENKO THEORY OF ELASTIC HOMOGENEOUS ISOTROPIC PRISMATIC BARS

Lecturer :
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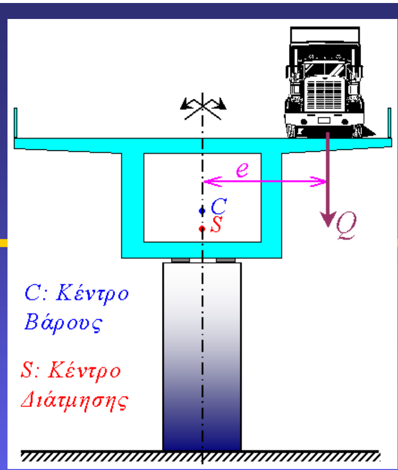
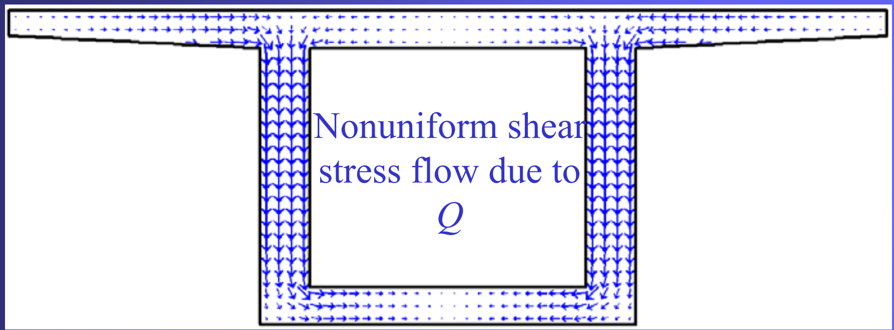
Professor NTUA

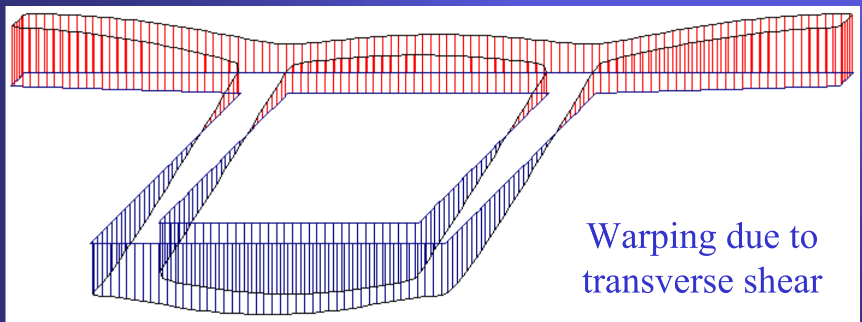
COURSE : APPLIED STRUCTURAL ANALYSIS OF FRAMED AND SHELL STRUCTURES (A1)

Eccentric Transverse Shear Loading *Q*



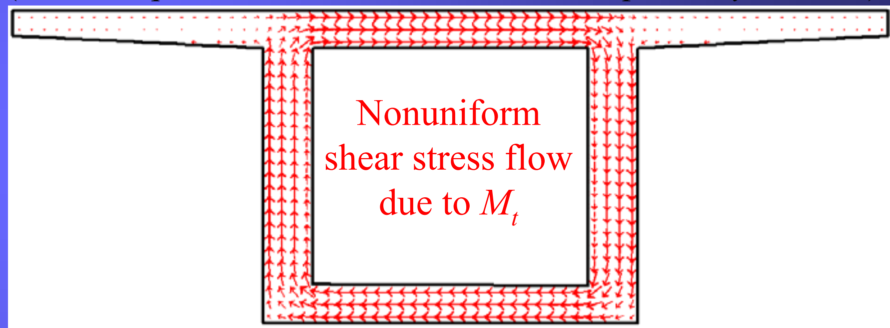
Shear Loading *Q*

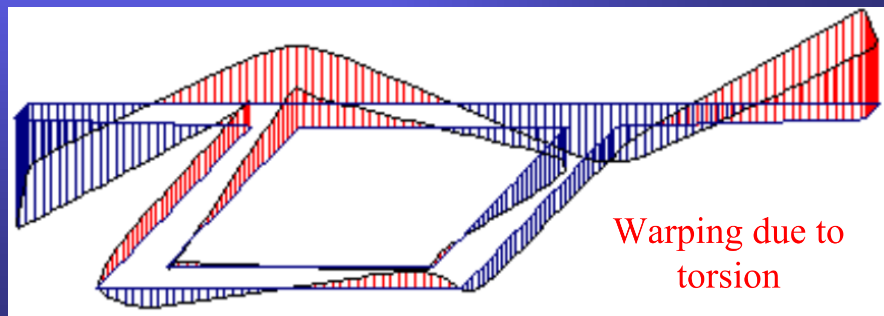




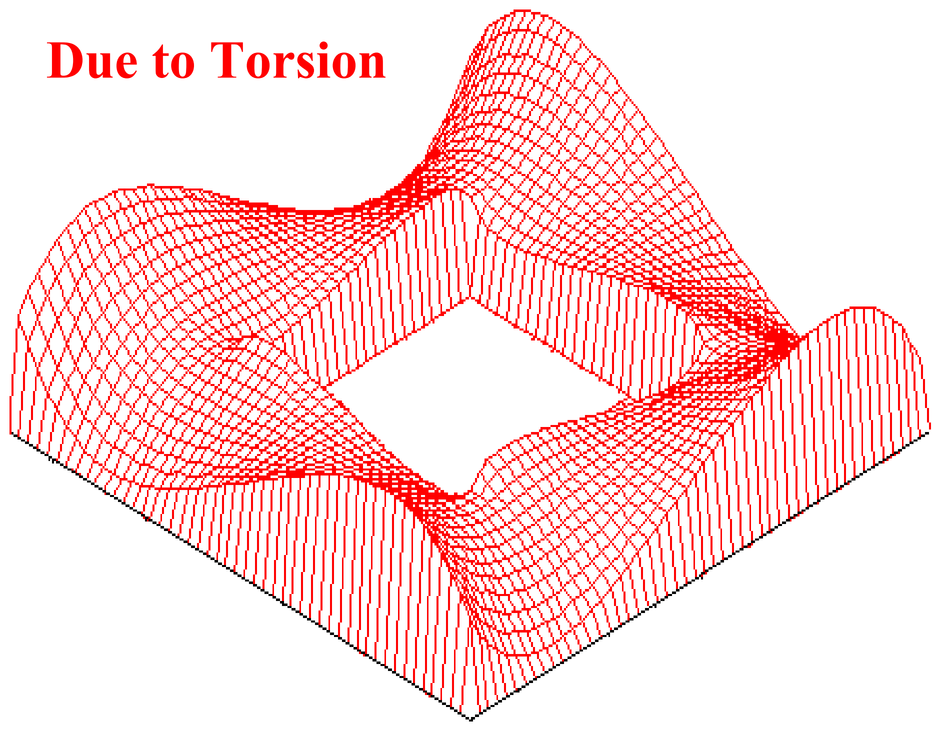
Direct Torsional Loading $M_t = Q \cdot e$)

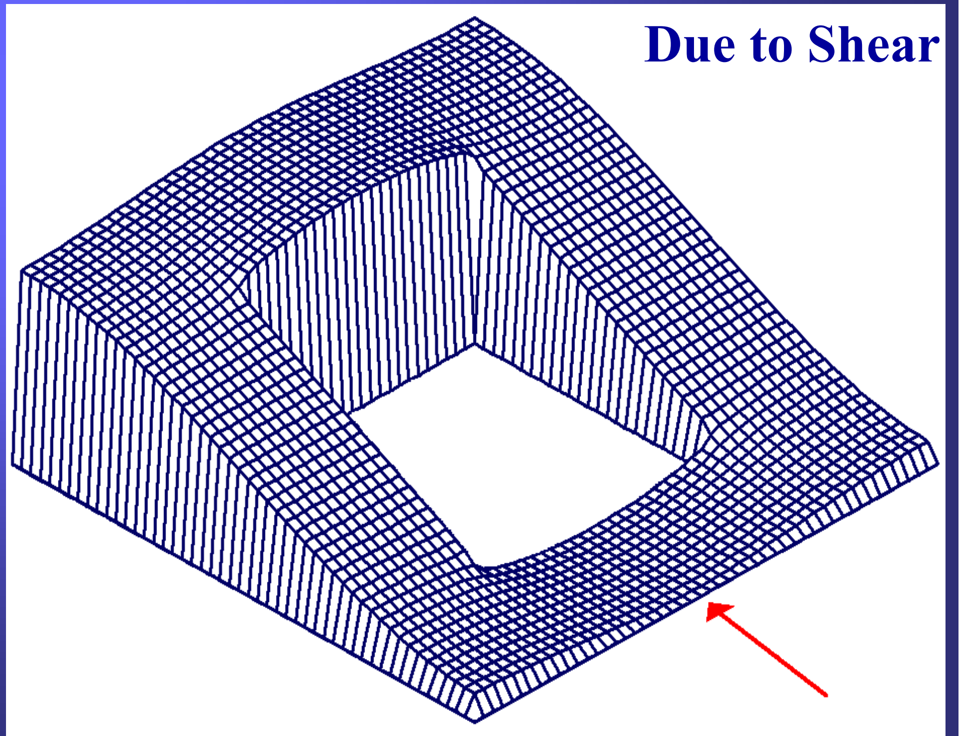
(***Direct***: *Equilibrium Torsion*, ***Indirect***: *Compatibility Torsion*)



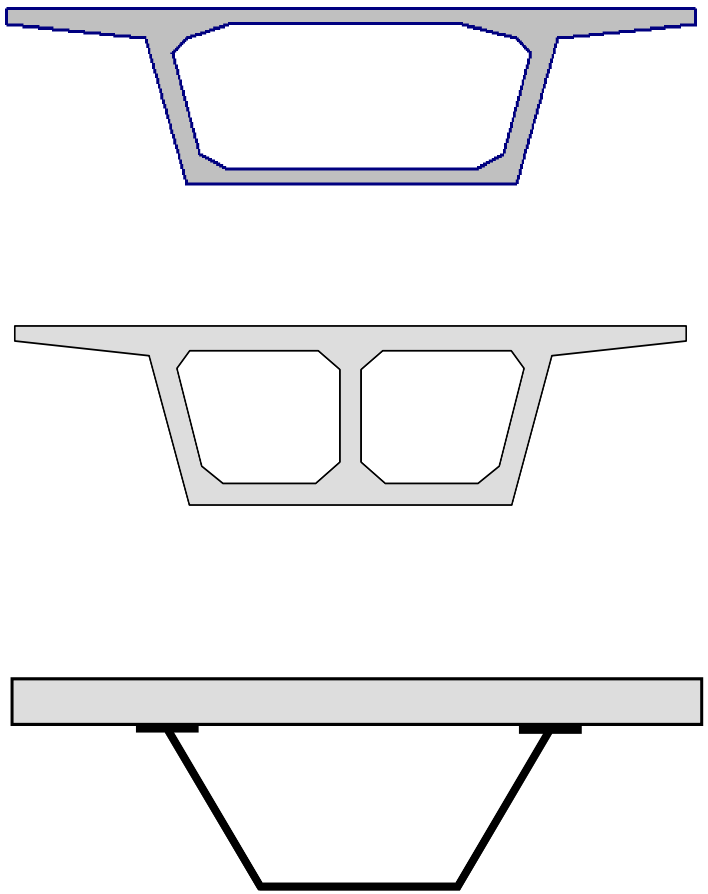


WARPING OF A HOLLOW SQUARE CROSS SECTION



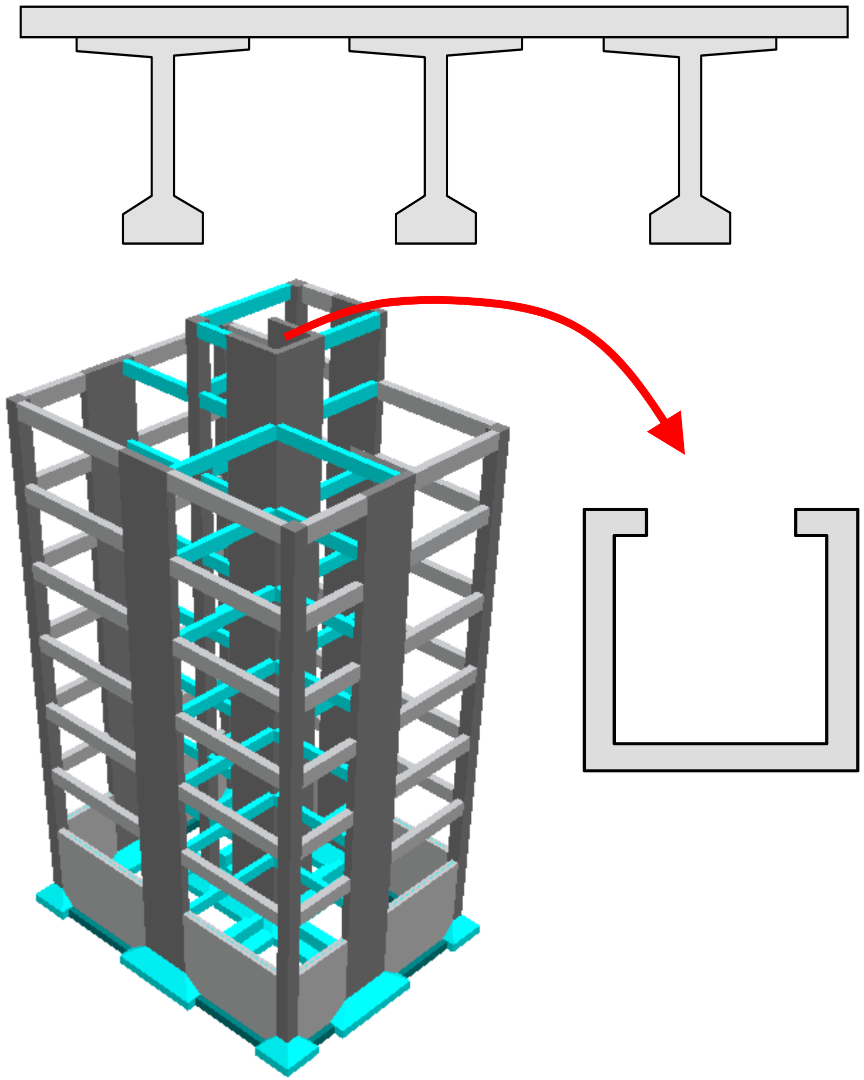


CROSS SECTIONS EXHIBITING SMALL AND SIGNIFICANT WARPING



SMALL WARPING

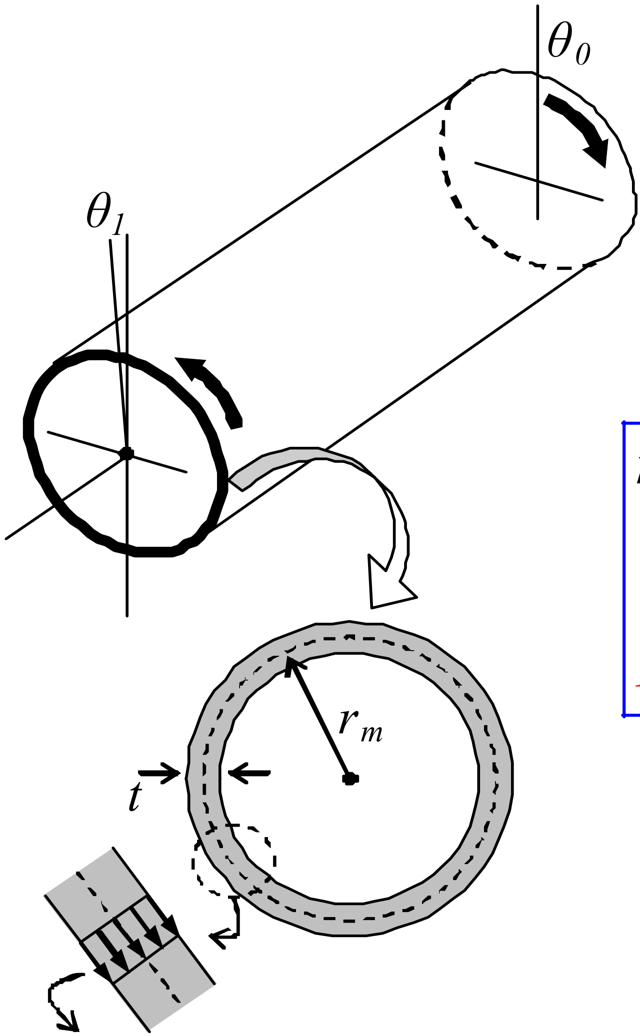
(Closed shaped cross sections)



INTENSE WARPING

(Open shaped cross sections)

COMPARISON OF TORSIONAL DEFORMATIONS OF THIN WALLED TUBES HAVING CLOSED AND OPEN SHAPED CROSS SECTIONS

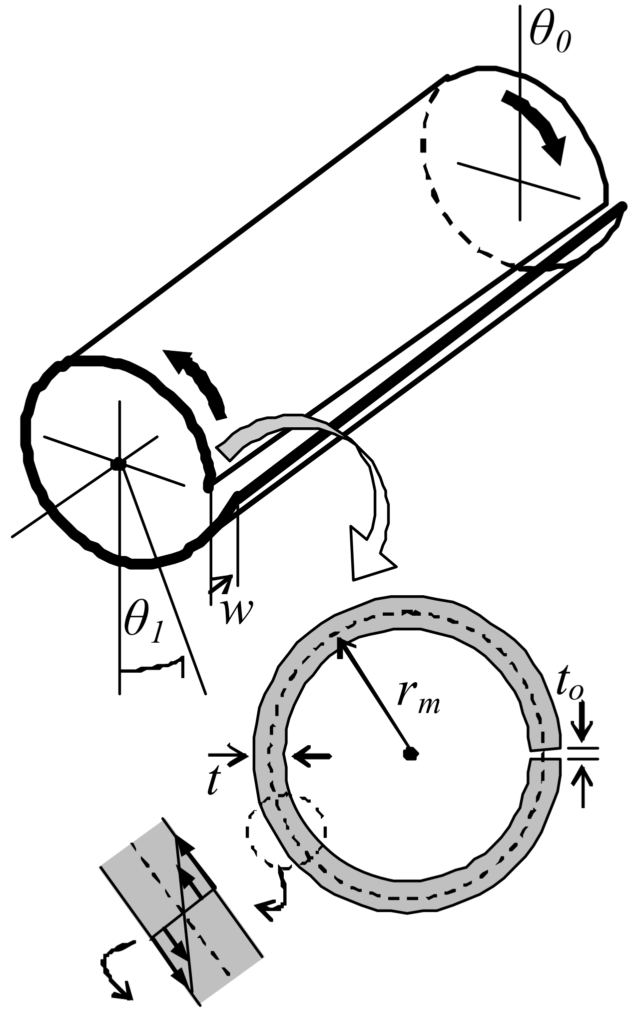


Distribution of primary shear stresses

$$r_m = 100mm$$

$$t = 1mm$$

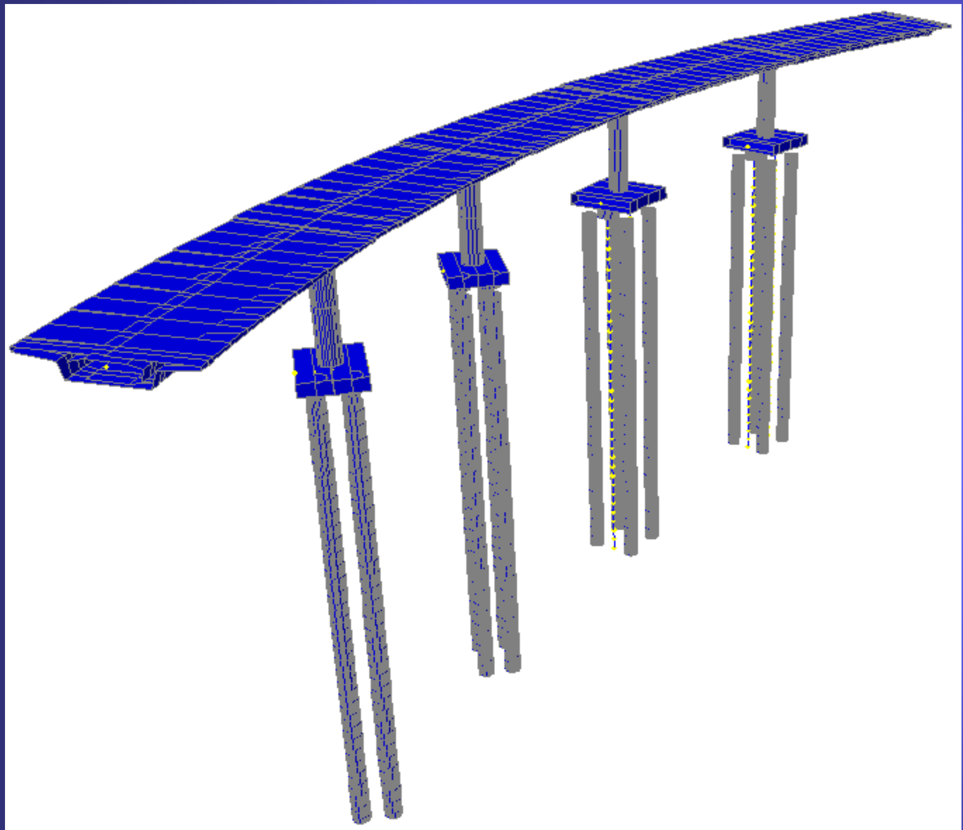
$$I_t^{Close} = 30000 I_t^{Open}$$



Distribution of primary shear stresses

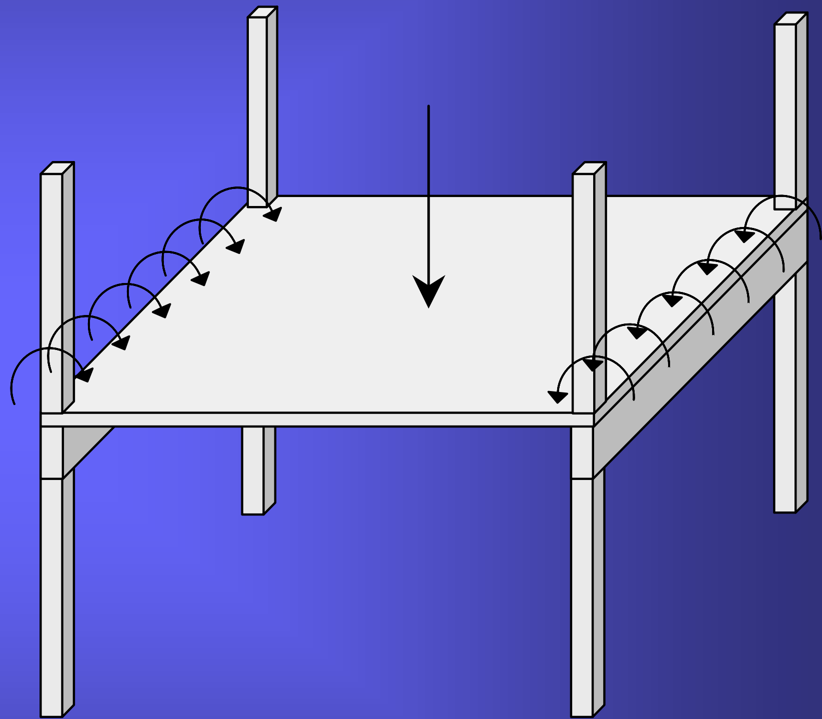
CLASSIFICATION OF TORSION AS A STRESS STATE

Direct Torsion (Equilibrium Torsion)



Bridge deck of box shaped cross section curved in plan → (Permanent) torsional loading due to self-weight

Indirect Torsion (Compatibility Torsion)



Cracking due to creep and shrinkage effects → Significant reduction of torsional rigidity

Classification of shear & torsion according to longitudinal variation of warping (UNIFORM - NONUNIFORM SHEAR AND TORSION)

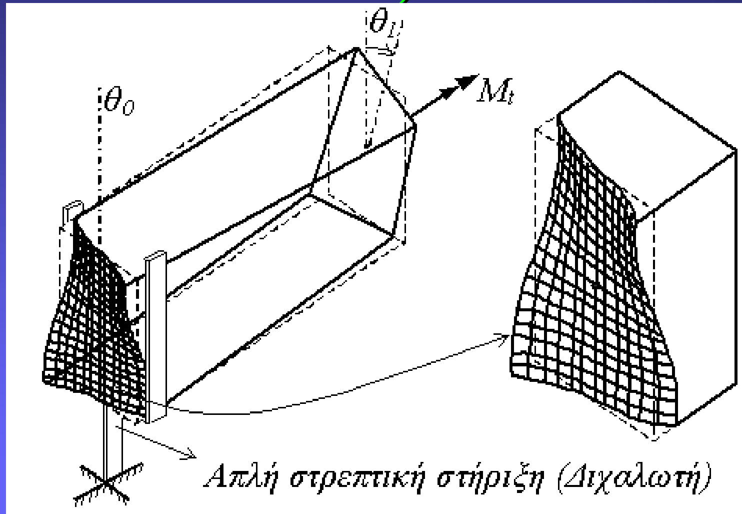
- **Transverse Load, Twisting Moment: Constant**
- **Warping (Q, M_t): Free (Not Restrained)**

↓

Uniform Shear – Torsion

Shear Stresses Exclusively

(Saint–Venant, 1855)



Ομοιόμορφη στρέψη ράβδου ορθογωνικής διατομής

- **Transverse Load, Twisting Moment: Variable**
- **Warping (Q, Mt): Restrained**

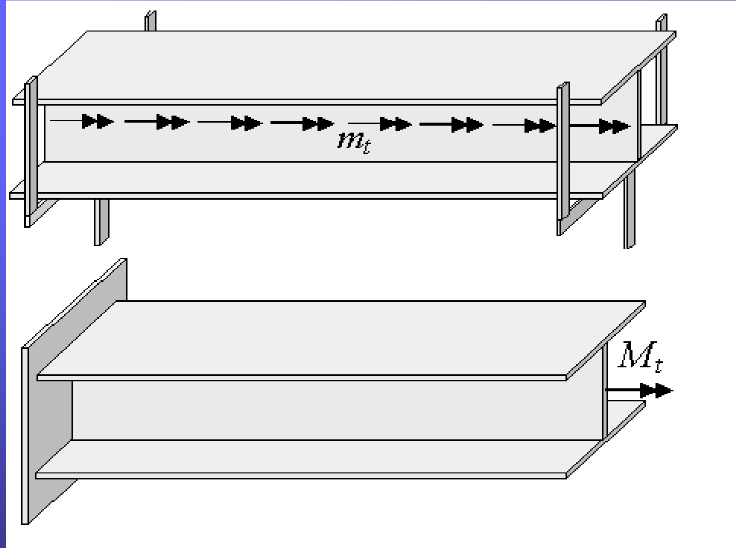
↓

Nonuniform Shear – Torsion

(Wagner, 1929)

Shear Stresses (Primary (St. Venant))

Stresses due to Warping (Normal stresses & Secondary shear stresses)

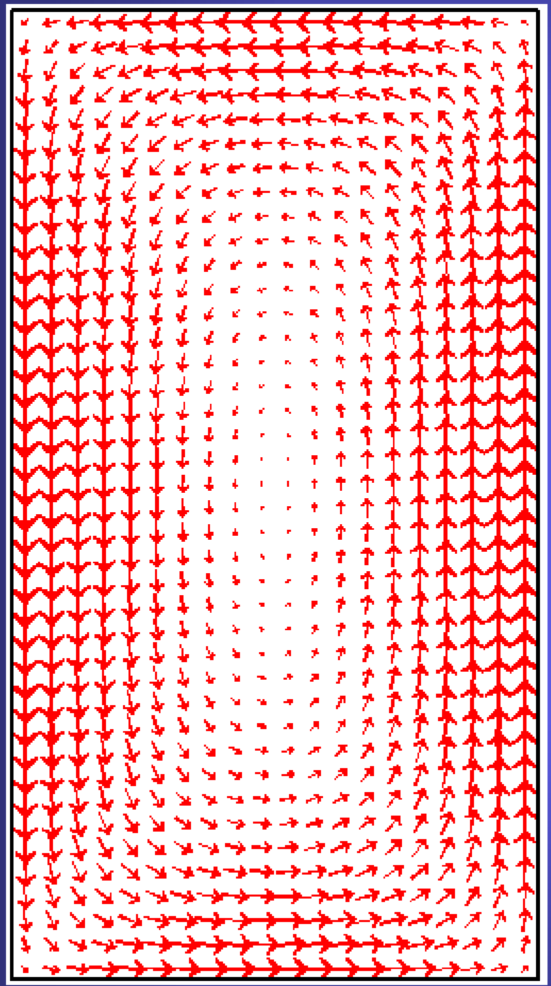


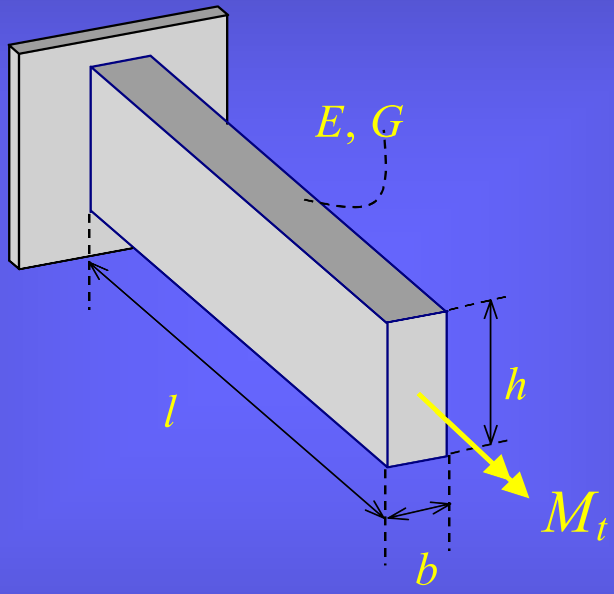
Ανομοιόμορφη στρέψη ράβδων διατομής διπλού "ταυ"

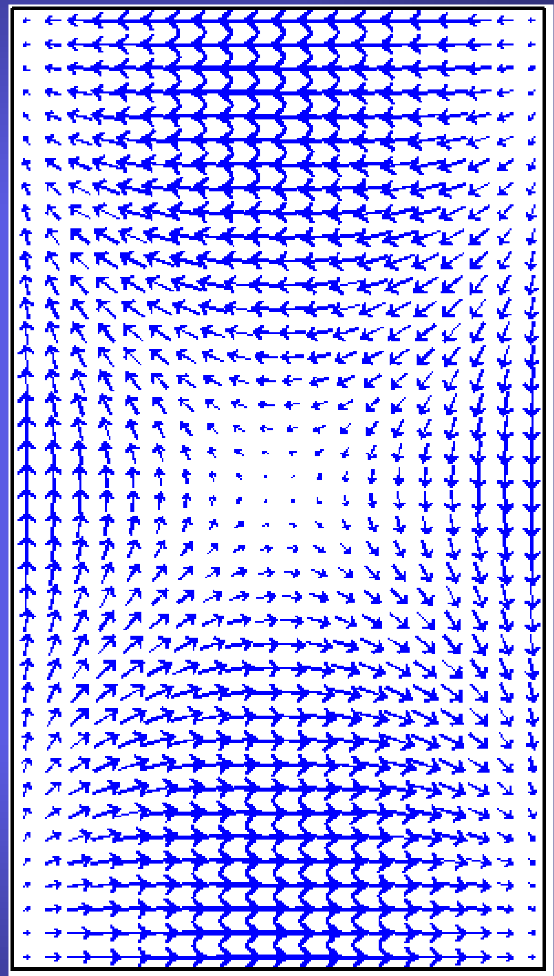
SHEAR STRESS DISTRIBUTION (NONUNIFORM TORSION)

Primary Shear Stresses (torsional loading)

Secondary (Warping) Shear stresses (torsional loading)





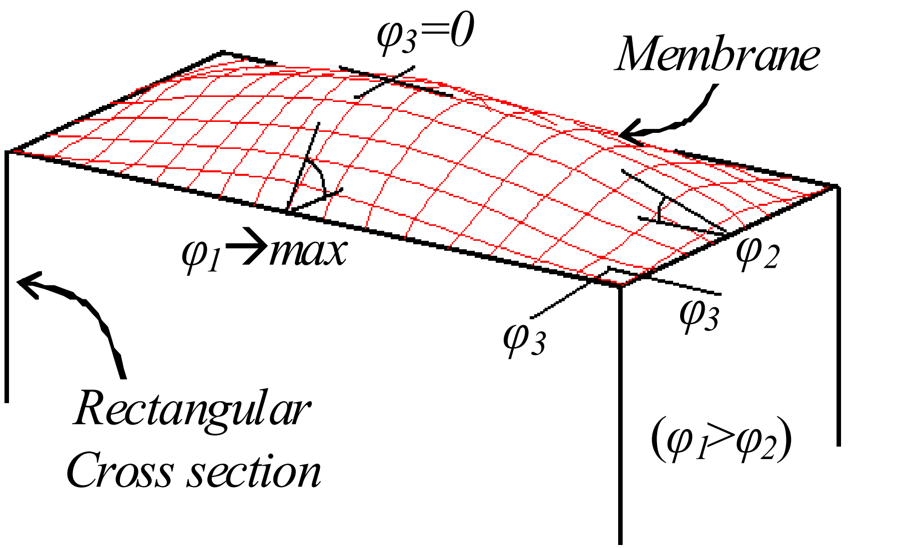


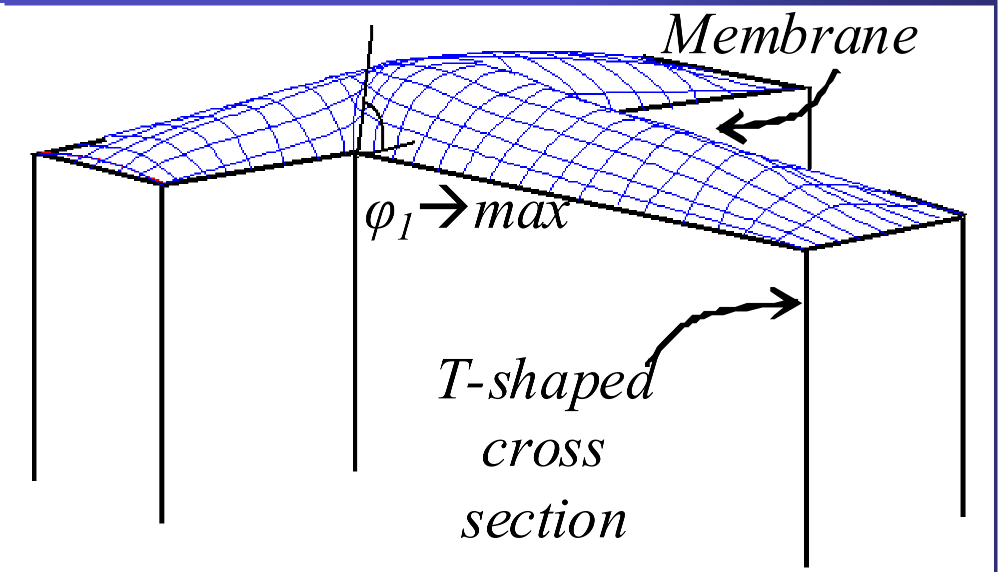
Closed Bredt stress flow, 1896

Complex distribution in thick walled cross sections (thin-walled: Vlasov, 1963)

PRANDTL'S MEMBRANE ANALOGY

Saint Venant (uniform) torsion has been "depicted" by Prandtl (1903) through the membrane analogy: Uniform torsion and membrane problems are described from analogous boundary value problems.

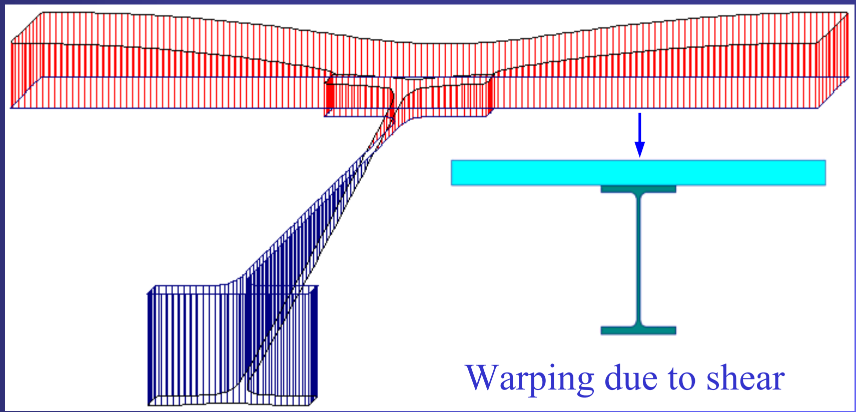




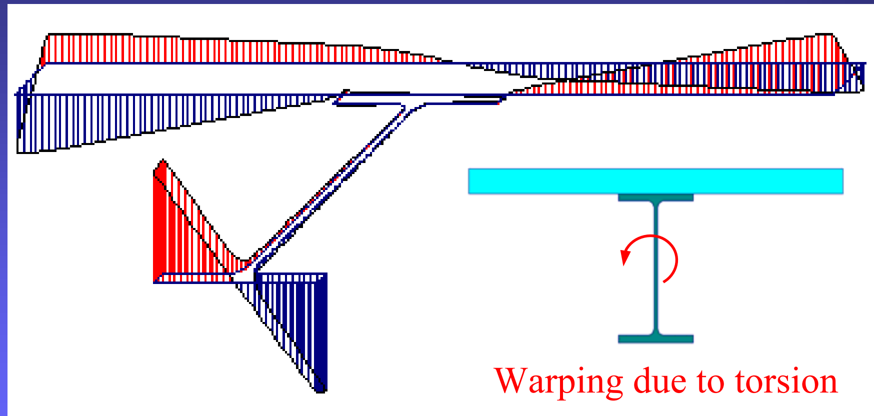
The deformed membrane offers the following information:

- **Contours correspond to the directions of the trajectories of shear stresses**
- **The slopes of the deformed membrane correspond to the values of shear stresses**
- **The volume of the deformed membrane corresponds to St. Venant's torsion constant**

SHEAR FORCE



TWISTING MOMENT



Warping due to shear << **Warping due to torsion**

(*Significant in Open Shaped Cross Sections*)

↓

- **Stress State (Stress field)**:

Uniform Shear

- **Strain/Deformation State:**

Shear Deformation Coefficients *Indirect account of warping deformation (Timoshenko, 1922)*

↓

- **Stress State**

&

- **Strain State:**

Nonuniform Torsion

Seven degrees of freedom (14x14 [K]*)*

- *Additional dof.: Twisting curvature*
- *Additional stress resultant: Warping moment*

Problem description of Nonuniform Torsion & Uniform Shear

Technical Beam Theory →

- *Limited set of cross sections (of simple geometry)*
- *Warping restraints are ignored*
- *Compatibility equations are not employed*
- *Stress computations are performed studying equilibrium of a finite segment of a bar and not equilibrium of an infinitesimal material point (3d elasticity)*

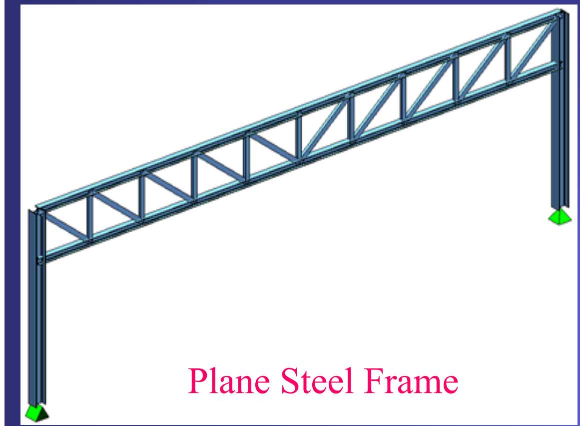
Thin Walled Beam theory (*Vlasov theory, 1964*) →

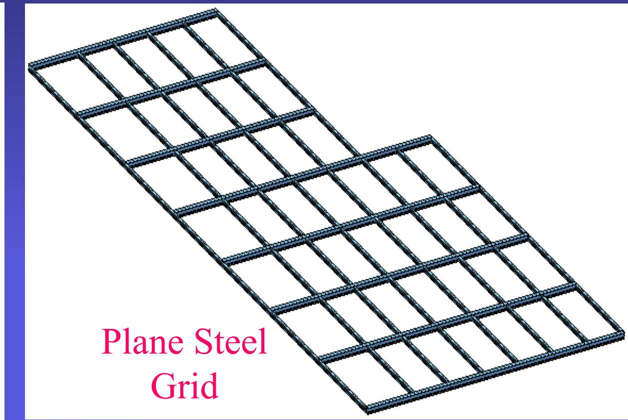
- *Valid for thin walled cross sections (Midline employed)*
- *Warping restraints are taken into account*
- *Reliability: Depends on thickness of shell elements comprising the beam*

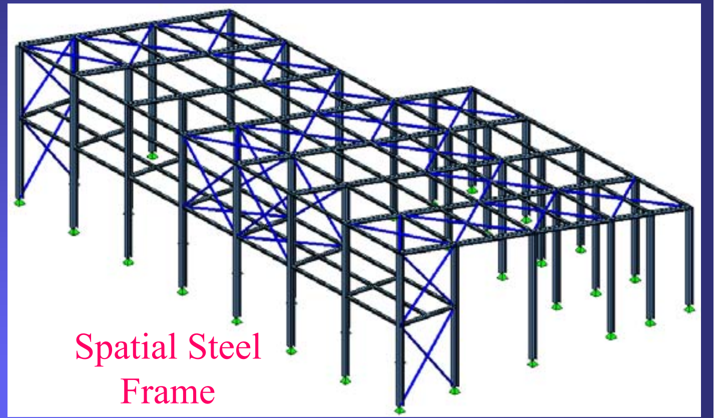
Generalized Beam Theory (*Schardt, 1966*) →

- *Valid for arbitrarily shaped cross sections (Thick or Thin walled)*
- *Warping restraints are taken into account*
- *BVPs formulated employing theory of 3D elasticity*
- *Numerical solution of BVPs*

Analysis of Bars and Bar Assemblages → Direct Stiffness Method







Everyday Engineering Practice:

- Application of 12x12 Stiffness Matrix (*6 dofs per node*)
- Approximate Computation of Torsion Constant
- Approximate Computation of Shear Deformation Coefficients
- Approximate Computation of stresses due to shear and torsion

↓

Inaccuracies → Non conservative Design (sometimes)

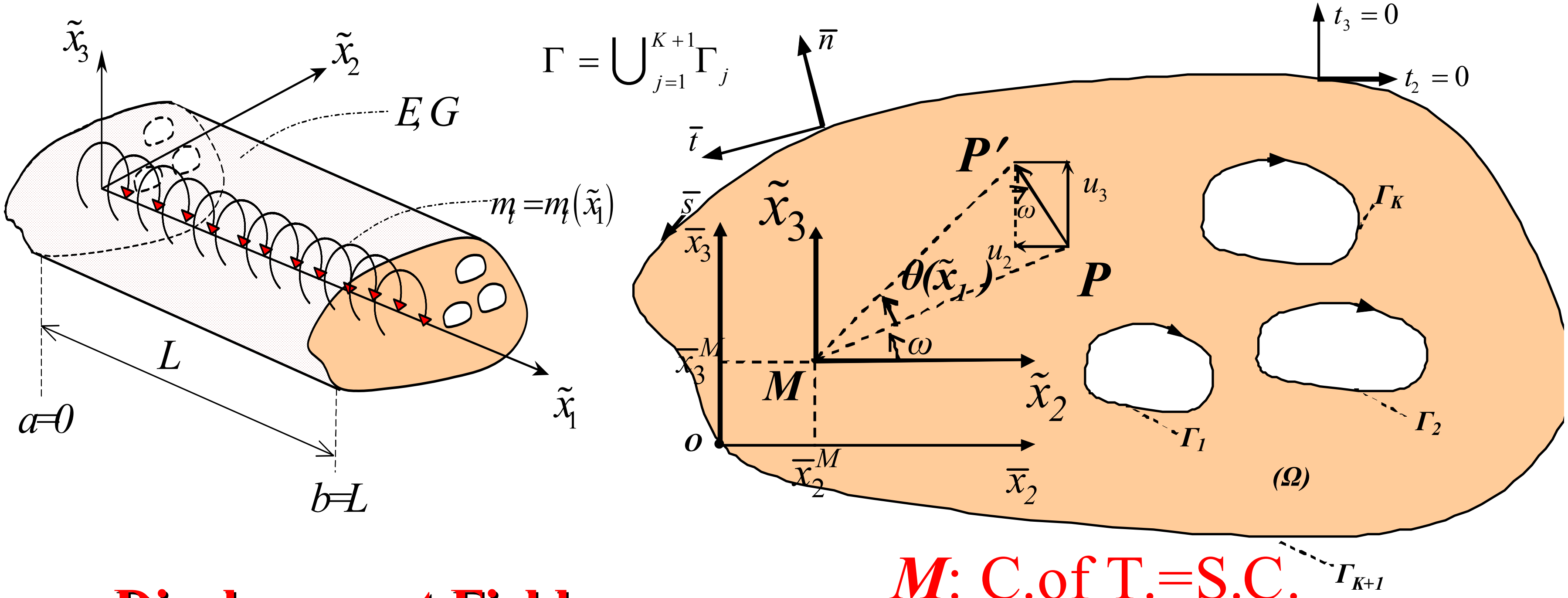
ASSUMPTIONS OF ELASTIC THEORY OF TORSION

- The bar is straight.
- The bar is prismatic.
- The bar's longitudinal axis is subjected to twisting exclusively.
- Distortional deformations of the cross section are not allowed (cross sectional shape is not altered during deformation ($\gamma_{23}=0$, *distortion neglected*).
- Twisting rotation is considered small: Circular arc displacements are approximated with the corresponding displacements along the chords.
- The material of the bar is homogeneous, isotropic, continuous (no cracking) and linearly elastic: Constitutive relations of linear elasticity are valid.
- The distribution of stresses at the bar ends is such so that all the aforementioned assumptions are valid.

Especially for **(unrestrained) uniform (Saint Venant) torsion** the following assumption is also valid:

- Longitudinal displacements (warping) are not restrained and do not depend on the longitudinal coordinate (every cross section exhibits the same warping deformations).

ELASTIC THEORY OF TORSION



***M*: C.of T.=S.C.**

(ν=0)

Displacement Field

$$u_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = +\theta_1'(\tilde{x}_1) \cdot \varphi_M(\tilde{x}_2, \tilde{x}_3)$$

$$u_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = -(PP')sin\omega = -(MP)\theta_1(\tilde{x}_1)\sin\omega = -\tilde{x}_3\theta_1(\tilde{x}_1)$$

$$u_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (PP')cos\omega = (MP)\theta_1(\tilde{x}_1)\cos\omega = \tilde{x}_2\theta_1(\tilde{x}_1)$$

ELASTIC THEORY OF TORSION

Components of the Infinitesimal Strain Tensor

$$\varepsilon_{11} = \frac{\partial u_1}{\partial \tilde{x}_1} = \theta_1''(\tilde{x}_1) \cdot \varphi_M(\tilde{x}_2, \tilde{x}_3) \qquad \varepsilon_{11} = 0 \rightarrow St.V.$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial \tilde{x}_2} = 0 \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial \tilde{x}_3} = 0 \qquad \gamma_{23} = \frac{\partial u_2}{\partial \tilde{x}_3} + \frac{\partial u_3}{\partial \tilde{x}_2} = 0$$

$$\gamma_{12} = \frac{\partial u_1}{\partial \tilde{x}_2} + \frac{\partial u_2}{\partial \tilde{x}_1} = \theta_1'(\tilde{x}_1) \cdot \left(\frac{\partial \varphi_M}{\partial \tilde{x}_2} - \tilde{x}_3 \right)$$

$$\gamma_{13} = \frac{\partial u_1}{\partial \tilde{x}_3} + \frac{\partial u_3}{\partial \tilde{x}_1} = \theta_1'(\tilde{x}_1) \left(\frac{\partial \varphi_M}{\partial \tilde{x}_3} + \tilde{x}_2 \right)$$

Components of the Cauchy Stress Tensor (ν=0)

$$\tau_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu(\varepsilon_{22}+\varepsilon_{33})\right] = E\cdot\theta_1''(\tilde{x}_1)\cdot\varphi_M(\tilde{x}_2,\tilde{x}_3)$$

$$\tau_{11} = 0 \rightarrow St.V.$$

$$\tau_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{22} + \nu(\varepsilon_{11}+\varepsilon_{33})\right] = 0$$

$$\tau_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11}+\varepsilon_{22})\right] = 0$$

$$\tau_{32} = G\cdot\gamma_{32} = 0$$

$$\tau_{12} = G\cdot\gamma_{12} = G\cdot\theta_1'(\tilde{x}_1)\cdot\left(\frac{\partial\varphi_M}{\partial\tilde{x}_2} - \tilde{x}_3\right) \qquad \tau_{31} = G\cdot\gamma_{31} = G\cdot\theta_1'(\tilde{x}_1)\cdot\left(\frac{\partial\varphi_M}{\partial\tilde{x}_3} + \tilde{x}_2\right)$$

ELASTIC THEORY OF TORSION

Differential Equilibrium Equations of 3D Elasticity

(Body forces neglected)

$$\left.\begin{aligned} G\cdot\theta_1''(\tilde{x}_1)\cdot\left(\frac{\partial\varphi_M}{\partial\tilde{x}_2}-\tilde{x}_3\right)=0 \\ G\cdot\theta_1''(\tilde{x}_1)\cdot\left(\frac{\partial\varphi_M}{\partial\tilde{x}_3}+\tilde{x}_2\right)=0 \end{aligned}\right\}\rightarrow$$

Not Satisfied! →

Inconsistency of Theory of Nonuniform Torsion:
Overall equilibrium of the bar is satisfied (energy principle). However, only the longitudinal equilibrium equation (along x_1) is satisfied locally (St.V. → Identical satisfaction of all diff. equil. eqns)

$$\underbrace{\frac{\partial}{\partial\tilde{x}_2}\left[G\cdot\theta_1'(\tilde{x}_1)\cdot\left(\frac{\partial\varphi_M}{\partial\tilde{x}_2}-\tilde{x}_3\right)\right]+\frac{\partial}{\partial\tilde{x}_3}\left[G\cdot\theta_1'(\tilde{x}_1)\left(\frac{\partial\varphi_M}{\partial\tilde{x}_3}+\tilde{x}_2\right)\right]+\frac{\partial}{\partial\tilde{x}_1}\left[E\cdot\theta_1''(\tilde{x}_1)\cdot\varphi_M\right]=0}$$

$$\downarrow$$

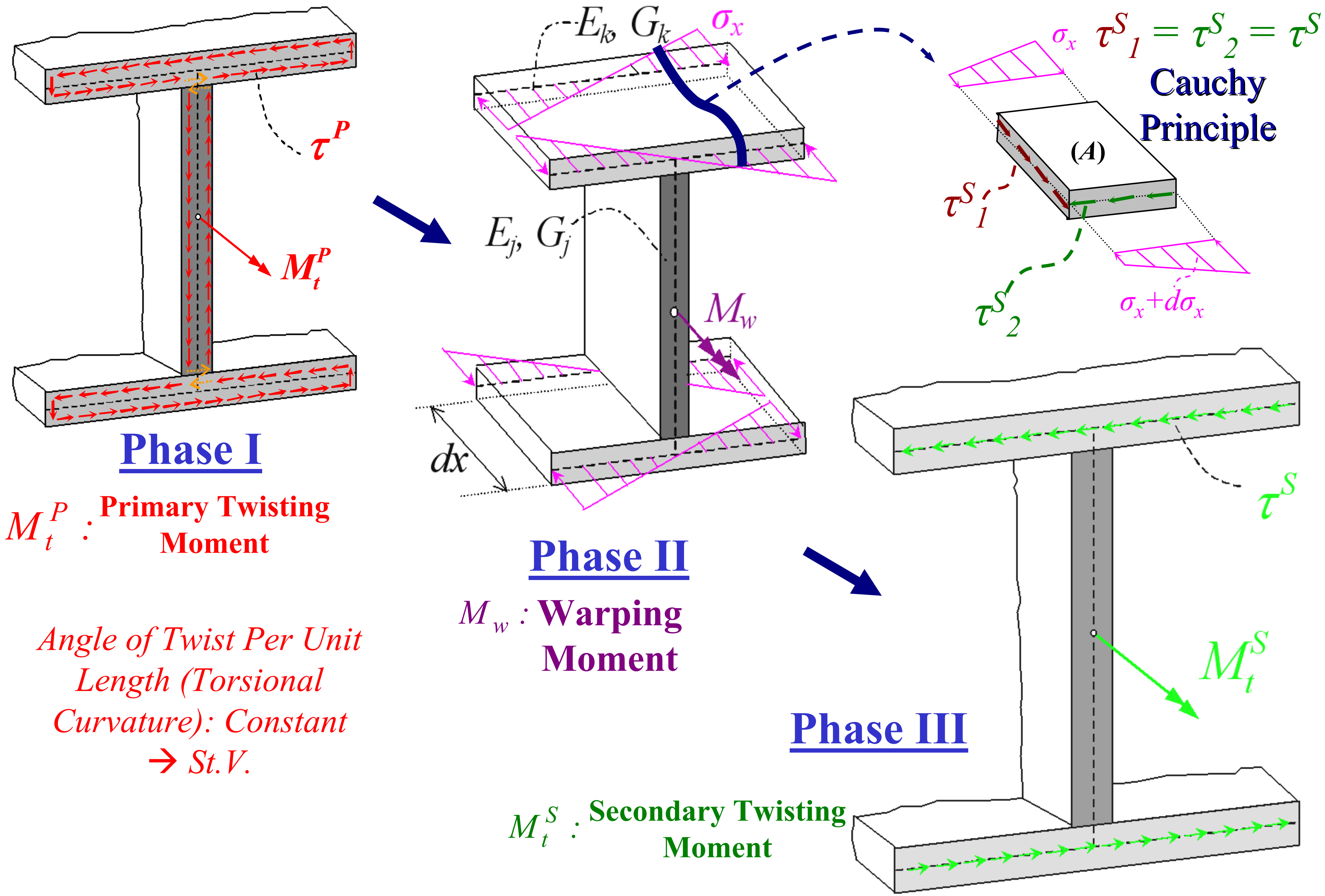
$$\frac{\partial^2\varphi_M}{\partial\tilde{x}_2{}^2}+\frac{\partial^2\varphi_M}{\partial\tilde{x}_3{}^2}=-\frac{E\cdot\theta_1'''(\tilde{x}_1)}{G\cdot\theta_1'(\tilde{x}_1)}\cdot\varphi_M$$

$$\varphi_M : \varphi_M\left(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\right)$$

$$\varepsilon_{11}=\theta_1''(\tilde{x}_1)\cdot\varphi_M\left(\tilde{x}_2,\tilde{x}_3\right)$$

Inconsistency → **DECOMPOSITION OF SHEAR STRESSES**

Physical Meaning of Decomposing Shear Stresses



ELASTIC THEORY OF TORSION

Primary (St. Venant) Shear Stresses

$$\tau_{12}^{P} = G \cdot \theta_1'(\tilde{x}_1) \cdot \left(\frac{\partial \varphi_M^P}{\partial \tilde{x}_2} - \tilde{x}_3 \right)$$

$$\tau_{13}^{P} = G \cdot \theta_1'(\tilde{x}_1) \cdot \left(\frac{\partial \varphi_M^P}{\partial \tilde{x}_3} + \tilde{x}_2 \right)$$

Secondary (Warping) Shear Stresses

$$\tau_{12}^{S} = G \cdot \frac{\partial \varphi_M^S}{\partial \tilde{x}_2}$$

$$\tau_{13}^{S} = G \cdot \frac{\partial \varphi_M^S}{\partial \tilde{x}_3}$$

Normal Stresses

$$\tau_{11}^{w} = E \cdot \theta_1''(\tilde{x}_1) \cdot \varphi_M^P(\tilde{x}_2, \tilde{x}_3)$$

$$\left(\frac{\partial \tau_{11}^P}{\partial \tilde{x}_2} + \frac{\partial \tau_{13}^P}{\partial \tilde{x}_3} \right) + \left(\frac{\partial \tau_{12}^S}{\partial \tilde{x}_2} + \frac{\partial \tau_{13}^S}{\partial \tilde{x}_3} + \frac{\partial \tau_{11}^w}{\partial \tilde{x}_1} \right) = 0$$

$$\frac{\partial \tau_{12}^P}{\partial \tilde{x}_2} + \frac{\partial \tau_{13}^P}{\partial \tilde{x}_3} = 0 \rightarrow \boxed{\nabla^2 \varphi_M^P = 0}$$

$$\frac{\partial \tau_{12}^S}{\partial \tilde{x}_2} + \frac{\partial \tau_{13}^S}{\partial \tilde{x}_3} + \frac{\partial \tau_{11}^w}{\partial \tilde{x}_1} = 0 \rightarrow$$

$$\boxed{\nabla^2 \varphi_M^S = -\frac{E \cdot \theta_1'''(\tilde{x}_1)}{G} \cdot \varphi_M^P}$$

Reliability of the shear stress decomposition

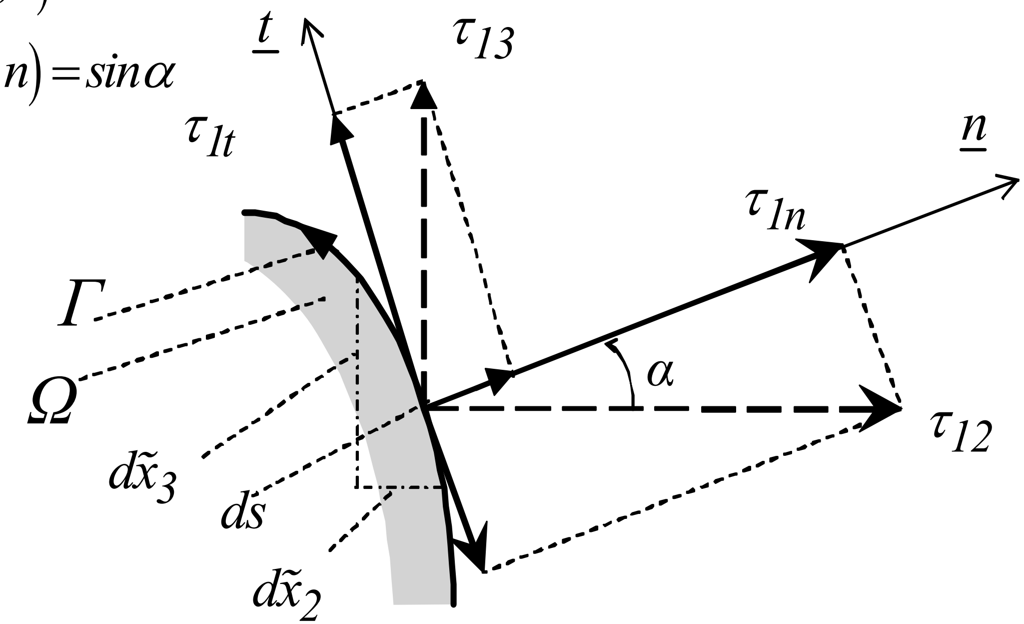
$$u_1^S \ll u_1^P = \theta_1'(\tilde{x}_1) \cdot \varphi_M^P(\tilde{x}_2, \tilde{x}_3)$$

ELASTIC THEORY OF TORSION

Boundary Conditions

$n_2 = cos(\tilde{x}_2, n) = cos\,\alpha$

$n_3 = sin(\tilde{x}_3, n) = sin\,\alpha$



Shear stresses along the normal direction ***n*** *on the boundary* ***VANISH:***

$$\tau_{1n}^{P} = \tau_{12}^{P} \cdot n_2 + \tau_{13}^{P} \cdot n_3 = 0$$

$$\tau_{1n}^{S} = \tau_{12}^{S} \cdot n_2 + \tau_{13}^{S} \cdot n_3 = 0$$

$$\tau_{1n}^{P} = G \cdot \theta_1'(\tilde{x}_1)\left(\frac{\partial \varphi_M^P}{\partial n} - \tilde{x}_3 \cdot n_2 + \tilde{x}_2 \cdot n_3\right) = 0 \rightarrow \frac{\partial \varphi_M^P}{\partial n} = \tilde{x}_3 \cdot n_2 - \tilde{x}_2 \cdot n_3$$

$$\tau_{1n}^{S} = G \cdot \frac{\partial \varphi_M^S}{\partial n} = 0 \rightarrow \frac{\partial \varphi_M^S}{\partial n} = 0$$

$$\tau_{1t}^{P} = G \cdot \theta_1'(\tilde{x}_1)\left(\frac{\partial \varphi_M^P}{\partial t} + \tilde{x}_2 \cdot n_2 + \tilde{x}_3 \cdot n_3\right) \qquad \tau_{1t}^{S} = G \cdot \frac{\partial \varphi_M^S}{\partial t}$$

ELASTIC THEORY OF TORSION

Boundary Value Problems

Primary Warping Function

Laplace differential eqn with Neumann type boundary conditions

$$\nabla^2 \varphi_M^P = \frac{\partial^2 \varphi_M^P}{\partial \tilde{x}_2^{\,2}} + \frac{\partial^2 \varphi_M^P}{\partial \tilde{x}_3^{\,2}} = 0 \, , \, \Omega$$

$$\frac{\partial \varphi_M^P}{\partial n} = \tilde{x}_3 \cdot n_2 - \tilde{x}_2 \cdot n_3 \, , \, \Gamma$$

Secondary Warping Function

Poisson differential eqn with Neumann type boundary conditions

$$\nabla^2 \varphi_M^S = \frac{\partial^2 \varphi_M^S}{\partial \tilde{x}_2^{\,2}} + \frac{\partial^2 \varphi_M^S}{\partial \tilde{x}_3^{\,2}} = -\frac{E \cdot \theta_1'''(\tilde{x}_1)}{G} \cdot \varphi_M^P \, , \, \Omega$$

$$\frac{\partial \varphi_M^S}{\partial n} = 0 \, , \, \Gamma$$

(PDEs)

ELASTIC THEORY OF TORSION

Stress Resultants

- **Twisting Moment:** $M_1 = M_1^P + M_1^S$

- **Primary Twisting Moment:**

$$M_1^P = \int_\Omega \left[\tau_{12}^P \left(\frac{\partial \varphi_M^P}{\partial \tilde{x}_2} - \tilde{x}_3 \right) + \tau_{13}^P \cdot \left(\frac{\partial \varphi_M^P}{\partial \tilde{x}_3} + \tilde{x}_2 \right) \right] d\Omega \rightarrow M_1^P = G \cdot I_t \cdot \theta_1'(\tilde{x}_1)$$

$$I_t = \int_\Omega \left(\tilde{x}_2^{\,2} + \tilde{x}_3^{\,2} + \tilde{x}_2 \cdot \frac{\partial \varphi_M^P}{\partial \tilde{x}_3} - \tilde{x}_3 \cdot \frac{\partial \varphi_M^P}{\partial \tilde{x}_2} \right) d\Omega$$

Torsion constant (*Saint-Venant*)

- **Secondary Twisting Moment:**

$$M_1^S = \int_\Omega \left(-\tau_{12}^S \frac{\partial \varphi_M^P}{\partial \tilde{x}_2} - \tau_{13}^S \frac{\partial \varphi_M^P}{\partial \tilde{x}_3} \right) d\Omega \rightarrow M_1^S = -E \cdot C_M \cdot \theta_1'''(\tilde{x}_1)$$

$$C_M = \int_\Omega \left(\varphi_M^P \right)^2 d\Omega$$

Warping Constant (*Wagner*)

ELASTIC THEORY OF TORSION

Stress Resultants

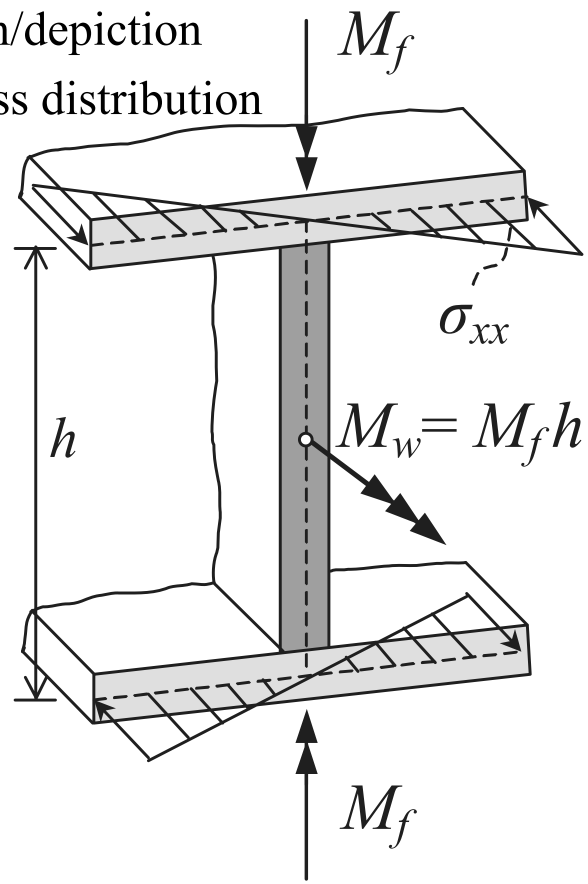
- **Warping Moment:** $M_W = -\int_{\Omega} \varphi_M^P \tau_{11}^w \, d\Omega \rightarrow M_W = -EC_M \theta_1''$

Schardt, 1966: "Higher order Stress Resultant"

Warping Moment as External Loading

Difficult visualization/depiction

Self-equilibrated stress distribution



- **Normal Stresses with Nonuniform Distribution**
- **Bending Moments applied in planes parallel to the longitudinal bar axis located at distance from the center of twist**
- **Concentrated Axial Forces:**

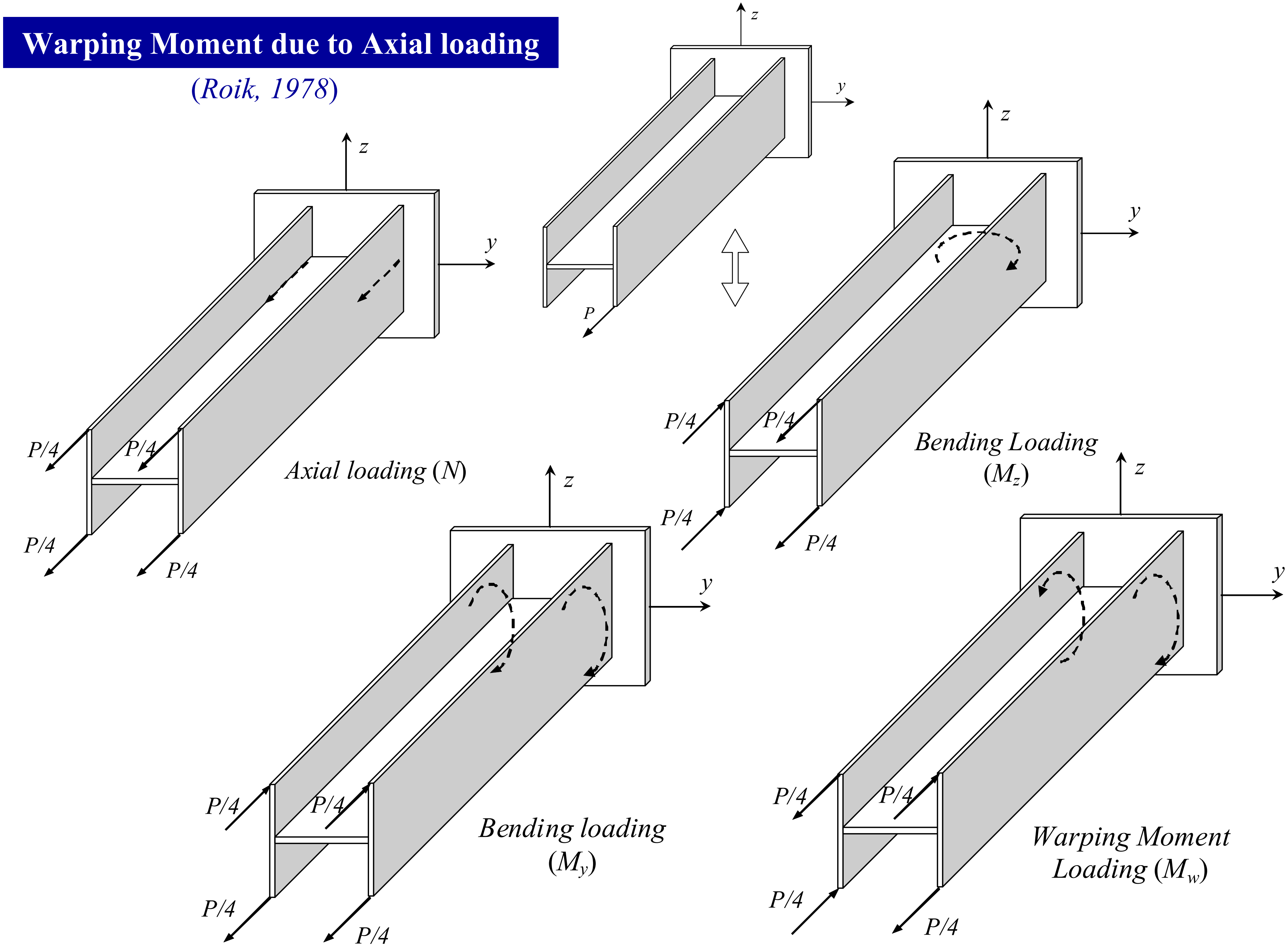
$$M_w = -\sum_{j=1}^{K} (P)_j \left(\varphi_M^P\right)_j$$

e.g. Z-shaped cross section with equal length flanges

→ $\varphi_M^P \neq 0$ *at centroid*

Warping Moment due to Axial loading

(*Roik, 1978*)



ELASTIC THEORY OF TORSION

Center of Twist (M)

$\bar{x}_2^M, \bar{x}_3^M$: Point with respect to which the cross sections rotate (*no transverse displacements*) (or point where rotation causes no axial and bending stress resultants

$\tau_{12}^P, \tau_{13}^P, I_t$: Independent of the center of twist (St. Venant could not calculate the position of the center of twist!)

$u_1^P, \tau_{12}^S, \tau_{13}^S, \tau_{11}^w, C_M$: Dependent of the center of twist

$$\varphi_M^P\left(\tilde{x}_2, \tilde{x}_3\right) = \phi_O^P(\bar{x}_2, \bar{x}_3) - \bar{x}_2\bar{x}_3^M + \bar{x}_3\bar{x}_2^M + \bar{c}$$

$$\nabla^2 \varphi_O^P = 0 \, , \Omega \qquad \frac{\partial \varphi_O^P}{\partial n} = \bar{x}_3 \cdot n_2 - \bar{x}_2 \cdot n_3 \, , \Gamma$$

- **Method of equilibrium**:

Under any coordinate system $N = M_2 = M_3 = 0$ due to warping normal stresses

- **Energy Method**:

Minimization of Strain Energy due to warping normal stresses

$$\frac{\partial C_M}{\partial \bar{x}_2} = \frac{\partial C_M}{\partial \bar{x}_3} = \frac{\partial C_M}{\partial \bar{c}} = 0$$

ELASTIC THEORY OF TORSION

Center of Twist (M)

$$\bar{S}_2\,\bar{x}_2^M - \bar{S}_3\,\bar{x}_3^M + A\,\bar{c} = -\bar{R}_S^P$$

$$\bar{I}_{22}\,\bar{x}_2^M + \bar{I}_{23}\,\bar{x}_3^M + \bar{S}_2\,\bar{c} = -\bar{R}_2^P$$

$$\bar{I}_{23}\,\bar{x}_2^M + \bar{I}_{33}\,\bar{x}_3^M - \bar{S}_3\,\bar{c} = \bar{R}_3^P$$

where:

$$A = \int_\Omega d\Omega \quad \bar{S}_2 = \int_\Omega \bar{x}_3\,d\Omega \quad \bar{S}_3 = \int_\Omega \bar{x}_2\,d\Omega$$

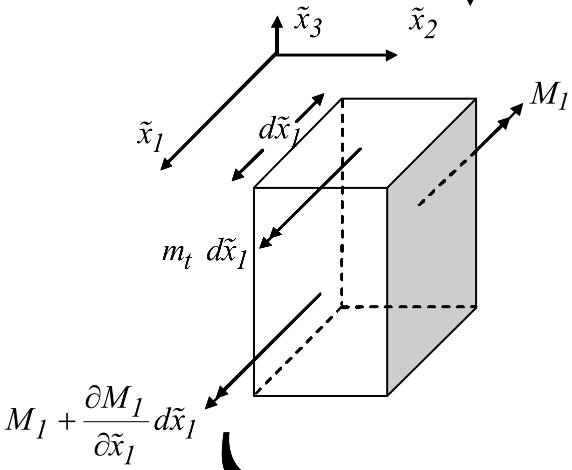
$$\bar{I}_{22} = \int_\Omega \bar{x}_3^{\,2} d\Omega \quad \bar{I}_{33} = \int_\Omega \bar{x}_2^{\,2} d\Omega \quad \bar{I}_{23} = -\int_\Omega \bar{x}_2 \bar{x}_3\,d\Omega$$

$$\bar{R}_S^P = \int_\Omega \varphi_O^P\,d\Omega \quad \bar{R}_2^P = \int_\Omega \bar{x}_3\,\varphi_O^P\,d\Omega \quad \bar{R}_3^P = \int_\Omega \bar{x}_2\,\varphi_O^P\,d\Omega$$

ELASTIC THEORY OF TORSION

Global Equilibrium Equation & Boundary conditions

Method of Equilibrium or Energy Method



TOTAL POTENTIAL ENERGY

$$\Pi_{o\lambda} = \int_0^L \underbrace{\left(\frac{1}{2} G \cdot I_t \cdot \theta_1'^2 + \frac{1}{2} E \cdot C_M \cdot \theta_1''^2 - m_t \cdot \theta_1 \right)}_{F} d\tilde{x}_1$$

$$\frac{\partial F}{\partial \theta_1} - \frac{d}{d\tilde{x}_1} \frac{\partial F}{\partial \theta_1'} + \frac{d^2}{d\tilde{x}_1^2} \frac{\partial F}{\partial \theta_1''} = 0 \quad (\textit{Euler–Lagrange eqns})$$

$$m_t = -G \cdot I_t \cdot \theta_1'' + E \cdot C_M \cdot \theta_1'''' \qquad \text{Inside the bar interval}$$

$$a_1\theta_1 + a_2 M_1 = a_3 \quad \beta_1\theta_1' + \beta_2 M_W = \beta_3 \qquad \text{At the bar ends}$$

Torsional Damping Coefficient

$$\varepsilon = L\sqrt{\frac{GI_t}{EC_M}} : \begin{cases} \geq 15 \rightarrow \text{Uniform Torsion} \\ < 15 \rightarrow \text{Nonuniform Torsion} \end{cases}$$

(*Ramm & Hofmann 1995*)

ELASTIC THEORY OF TORSION

Alternative Solution of the Uniform Torsion Problem

• ***Conjugate function*** ψ ***of function*** φ_M^P

$$\nabla^2\psi = \left(\frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}\right) = 0 \qquad \tau_{12}^P = G\theta' \cdot \left(\frac{\partial\psi}{\partial x_3} - x_3\right) \quad \tau_{13}^P = G\theta' \cdot \left(-\frac{\partial\psi}{\partial x_2} + x_2\right)$$

$$\psi = \frac{1}{2}\cdot\left(x_2^2 + x_3^2\right) + C_\psi \qquad I_t = \int_\Omega \left(x_2^2 + x_3^2 - x_2\cdot\frac{\partial\psi}{\partial x_2} - x_3\cdot\frac{\partial\psi}{\partial x_3}\right) d\Omega$$

• ***Prandtl Stress function F(x,y)***

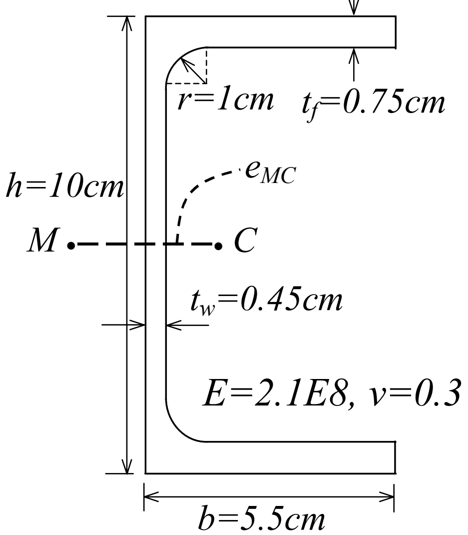
$$\nabla^2 F = \left(\frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_3^2}\right) = 1 \qquad \tau_{12}^P = -2\cdot G\theta' \cdot \frac{\partial F}{\partial x_3} \quad \tau_{13}^P = 2\cdot G\theta' \cdot \frac{\partial F}{\partial x_2}$$

$$F = C_F \qquad I_t = \int_\Omega \left(x_2\cdot\frac{\partial F}{\partial x_2} + x_3\cdot\frac{\partial F}{\partial x_3}\right) d\Omega$$

Constants C_ψ, C_F are unknown and must be determined at each boundary of a multiply connected region (occupied by the cross section) → Complex Problem.

Example

Steel Profile UPE–100

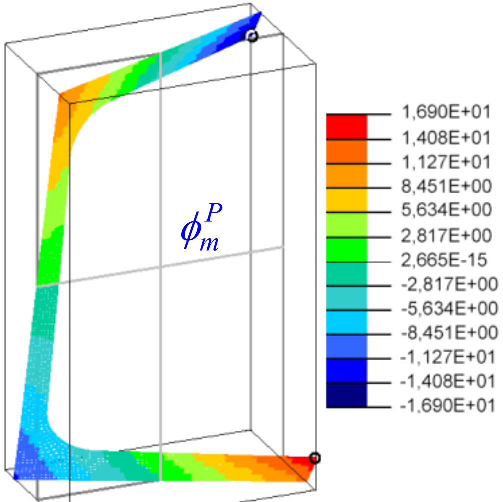


- Bar ends simply supported
- Loading: $m_t = 1kNm/m$
- Length: $l=1.0$m

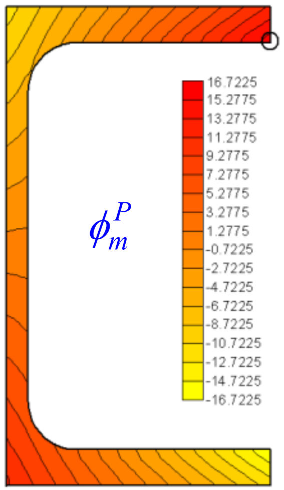
$\varepsilon = 3.66 < 15$

Στρεπτικά μεγέθη	BEM	FEM [Kraus, 2005]	Πίνακες (ΘΛΡ) [Schneider, 2001]	Απόκλιση (%) BEM & Πίνακες
$e_{MC}(cm)$	3.754	3.758	3.925	4.55
$I_t(cm^4)$	2.019	2.010	1.995	1.23
$C_M(cm^6)$	590.2	590.1	568.1	3.75
$max\,\phi_M^P(cm^2)$	16.73	16.90	14.02	**16.16** $\rightarrow(\sigma_w)$

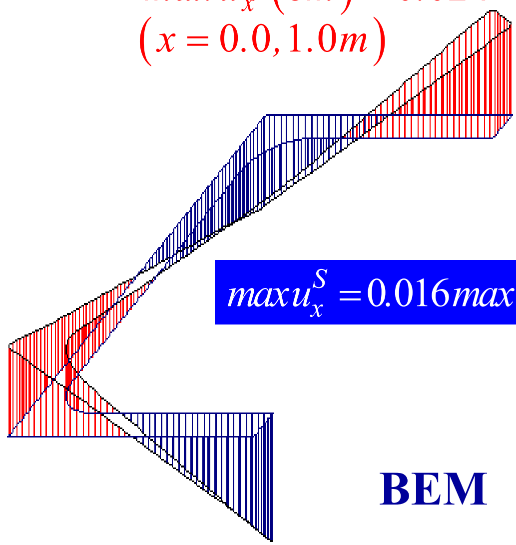
FEM (Kraus, 2005)



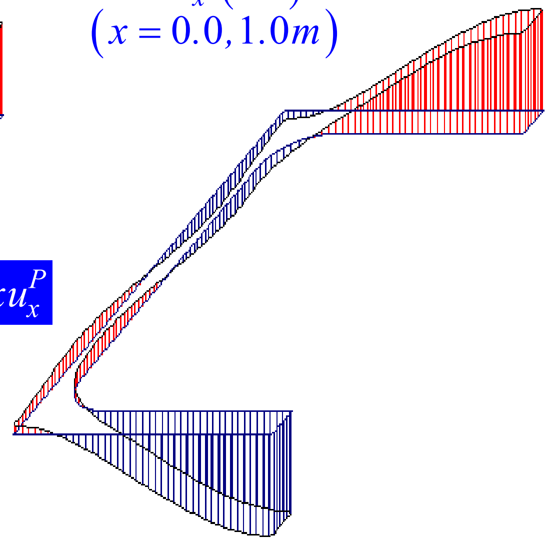
BEM



$max\,u_x^P(cm) = 0.024$
$(x = 0.0, 1.0m)$



$max\,u_x^S(cm) = 4.052\times10^{-4}$
$(x = 0.0, 1.0m)$



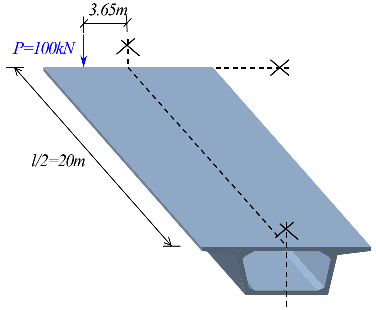
$max\,u_x^S = 0.016\,max\,u_x^P$

BEM

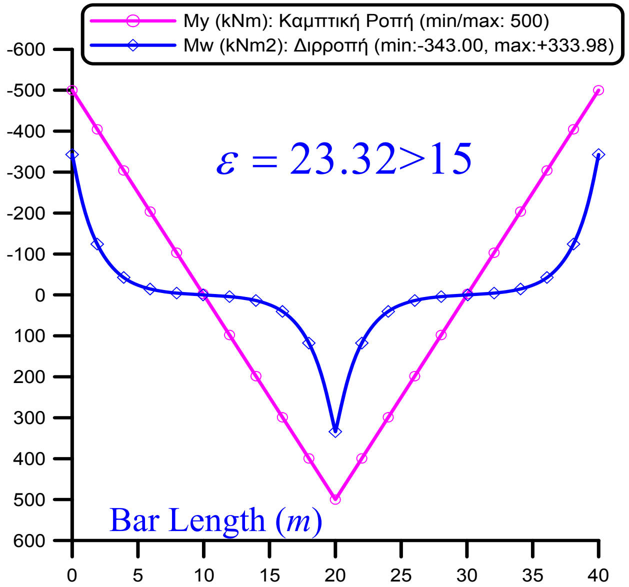
Warping along the thickness direction IS NOT CONSTANT

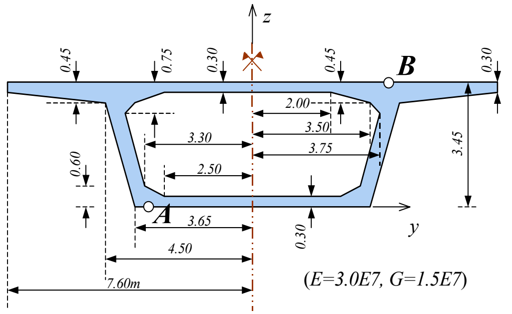
→ ***Thin walled beam theory not valid***

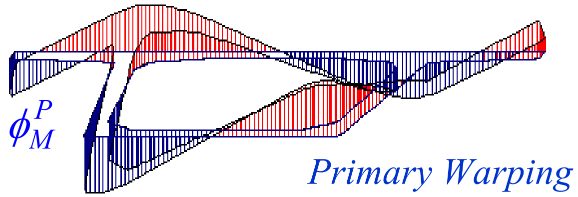
Example



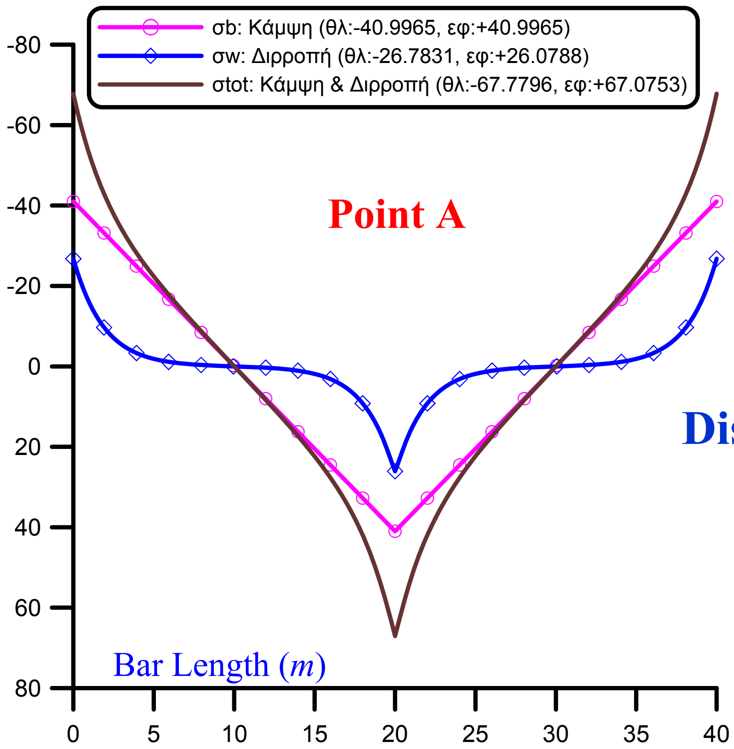
Bar of Box Shaped Cross Section Clamped at Both Ends



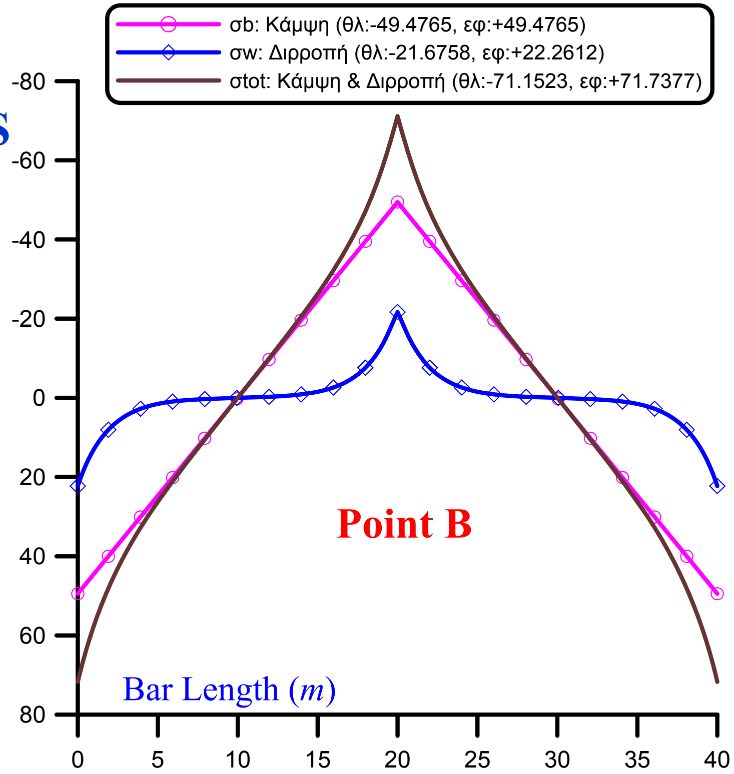




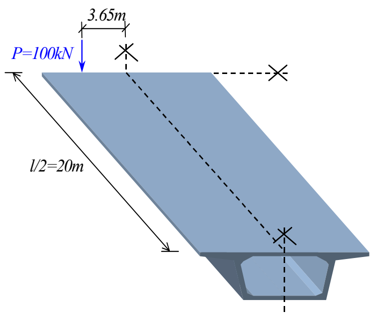
NORMAL STRESSES



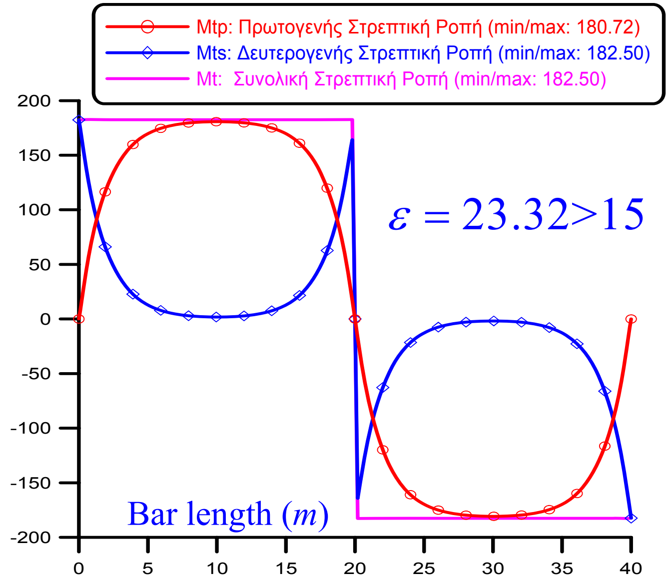
Discrepancy $\cong 30\%$

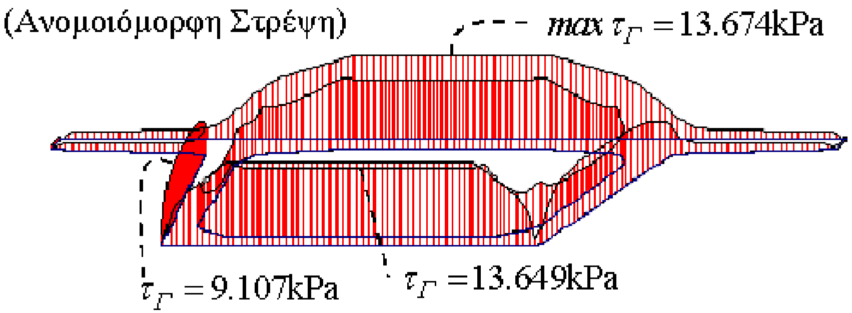


Example

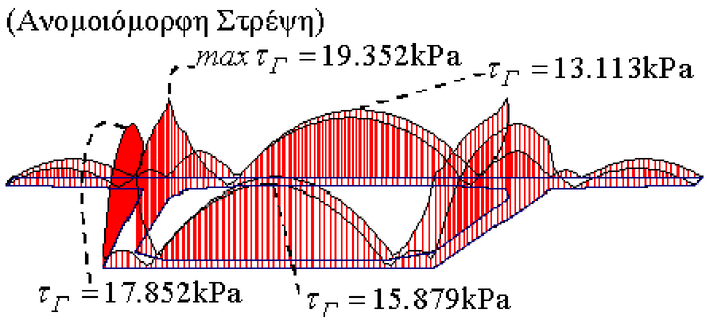


Bar of Box Shaped Cross Section Clamped at Both Ends



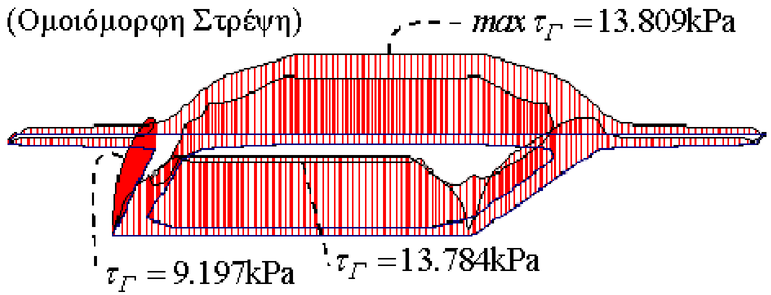


$max\, M_t^P = 180.72 kNm$ στις θέσεις $x = 10.0, 30.0m$



$max\, M_t^S = 182.5 kNm$ στις θέσεις $x = 0.0, 40.0m$

SHEAR STRESSES



$max\, M_t = 182.5 kNm$ σ΄ όλη τη ράβδο

Discrepancy $\cong 30\%$

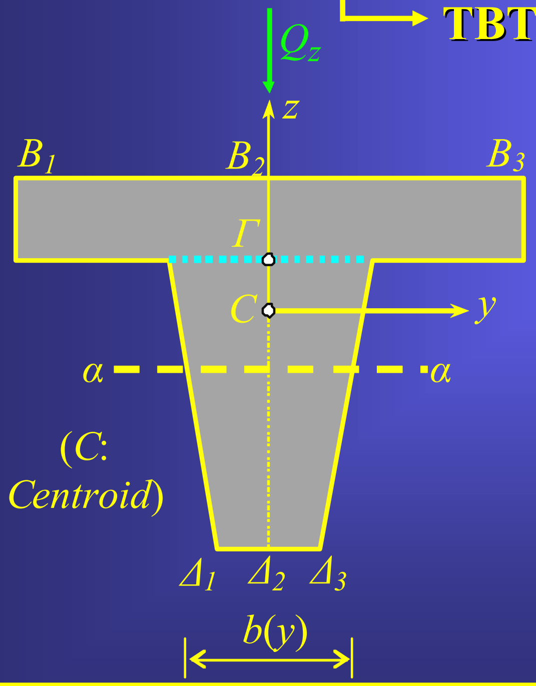
UNIFORM SHEAR BEAM THEORY

- Computation of Shear Stresses
- Computation of Shear Center Position
- Computation of Shear Deformation Coefficients (required for Timoshenko beam theory)

Shear Stresses
- → **GBT** → **BVP**
  - → *Displacement Field*
  - → *Stress Field* → *Poisson ratio ν taken into account*
- → **TWBT**
- → **TBT** → Q_z :

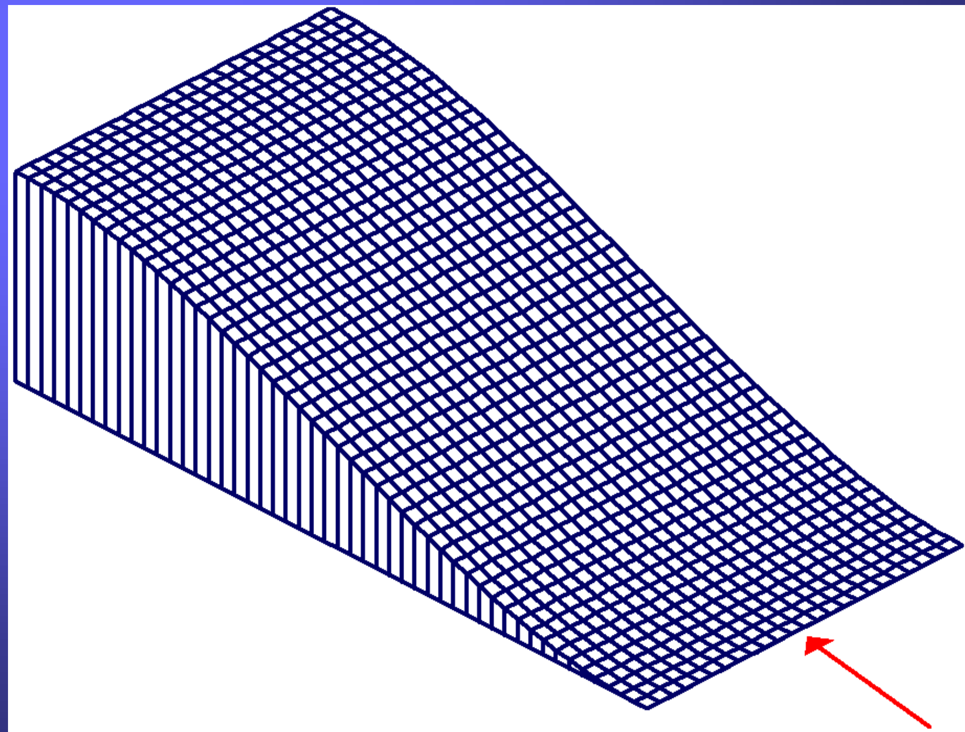
$$\begin{cases} \tau_{xz} = \dfrac{Q_z S_y^{cut}}{I_{yy} b} \\ \tau_{xy} = 0 \end{cases}$$

- → *Cross sections possessing at least one axis of symmetry*
- → *Simply connected cross section*
- → *Poisson ratio ν neglected*
- → τ_{xz}*: constant along the width b*
- → B_1B_3 & $\Delta_1\Delta_3$ → τ_{xz}*: vanishing*
- → $B_2\Delta_2$ *point Γ* → τ_{xz}*: Discontinuity*

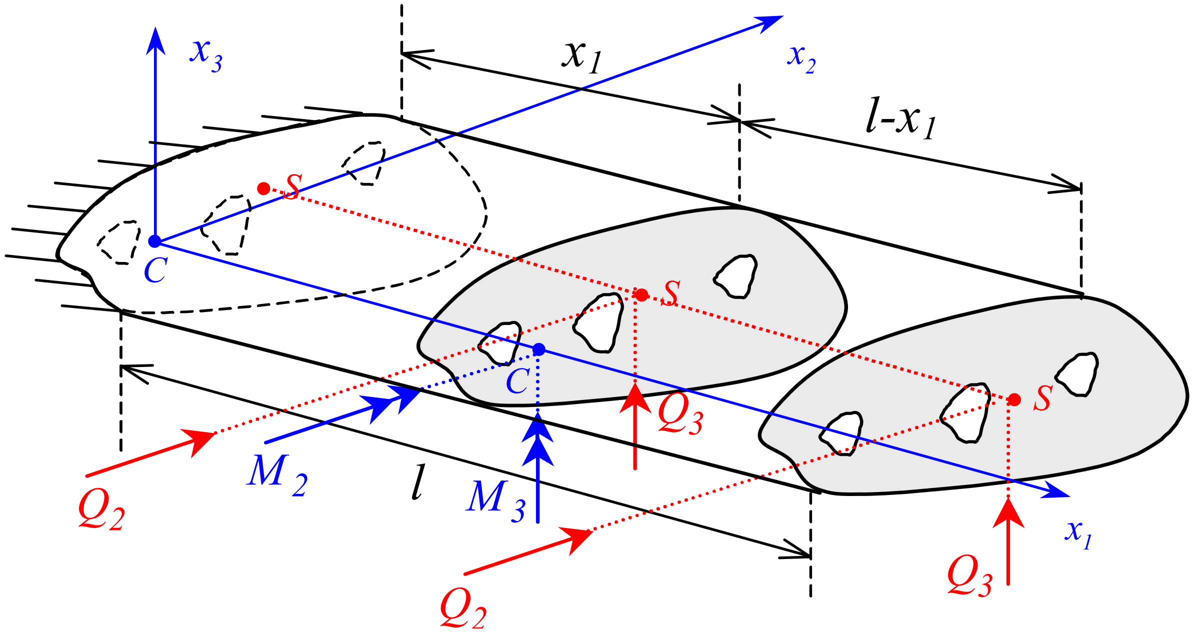


ASSUMPTIONS OF ELASTIC THEORY OF UNIFORM SHEAR

- The bar is straight.
- The bar is prismatic.
- Distortional deformations of the cross section are not allowed ($\gamma 23=0$, distortion neglected).
- The material of the bar is homogeneous, isotropic, continuous (no cracking) and linearly elastic: Constitutive relations of linear elasticity are valid.
- The distribution of stresses at the bar ends is such so that all the aforementioned assumptions are valid.
- Deflections and bending rotations are considered to be small (geometrically linear theory).
- Longitudinal displacements (warping) are not restrained and do not depend on the longitudinal coordinate (every cross section exhibits the same warping deformations).



THEORY OF UNIFORM SHEAR – DISPLACEMENT FIELD



Displacement field

$$u_1(x_1,x_2,x_3)=\theta_2(x_1)\cdot x_3-\theta_3(x_1)x_2+\boxed{\varphi_c(x_2,x_3)}$$

$$u_2(x_1,x_2,x_3)=u_2(x_1)$$

$$u_3(x_1,x_2,x_3)=u_3(x_1)$$

By ignoring Shear Strains:

$$\theta_2(x_1)=-\frac{\partial u_3}{\partial x_1}\qquad \theta_3(x_1)=\frac{\partial u_2}{\partial x_1}$$

Components of the Infinitesimal Strain Tensor

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial \theta_2}{\partial x_1} x_3 - \frac{\partial \theta_3}{\partial x_1} x_2 \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0 \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2}\left(-\theta_3 + \frac{\partial u_2}{\partial x_1} + \frac{\partial \varphi_c}{\partial x_2} \right) = \frac{1}{2}\frac{\partial \varphi_c}{\partial x_2}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) = 0$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2}\left(\theta_2 + \frac{\partial \varphi_c}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2}\frac{\partial \varphi_c}{\partial x_3}$$

Components of the Cauchy Stress Tensor (v=0)

$$\tau_{11} = E \cdot \varepsilon_{11} = E\left(\frac{\partial \theta_2}{\partial x_1} x_3 - \frac{\partial \theta_3}{\partial x_1} x_2\right) \qquad \tau_{22} = 0 \quad \tau_{33} = 0 \quad \tau_{23} = 0$$

$$\tau_{12} = 2G \cdot \varepsilon_{12} = G \frac{\partial \varphi_c}{\partial x_2} \qquad \tau_{13} = 2G \cdot \varepsilon_{13} = G \frac{\partial \varphi_c}{\partial x_3}$$

Differential Equilibrium Equations of 3D Elasticity

(*Body forces neglected*)

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} = 0 \rightarrow \boxed{E\left(\theta_2'' \cdot x_3 - \theta_3'' \cdot x_2\right)} + G\frac{\partial^2 \varphi_c}{\partial {x_2}^2} + G\frac{\partial^2 \varphi_c}{\partial {x_3}^2} = 0$$

$$\left.\begin{aligned} \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} = 0 \rightarrow \frac{\partial \tau_{12}}{\partial x_1} = 0 \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} = 0 \rightarrow \frac{\partial \tau_{13}}{\partial x_1} = 0 \end{aligned}\right\} : \quad \text{Identical Satisfaction}$$

Determination of: $E\left(\theta_2''\cdot x_3-\theta_3''\cdot x_2\right)$

$$\left.\begin{aligned}
M_2 &= \int_\Omega \tau_{11}x_3\cdot d\Omega = E\left(\theta_2'\int_\Omega x_3{}^2 d\Omega-\theta_3'\int_\Omega x_2x_3 d\Omega\right)=E\left(\theta_2'\cdot I_{22}-\theta_3'\cdot I_{23}\right)\\
M_3 &= -\int_\Omega \tau_{11}x_2\cdot d\Omega = -E\left(\theta_2'\int_\Omega x_2x_3 d\Omega-\theta_3'\int_\Omega x_2{}^2 d\Omega\right)=E\left(\theta_3'\cdot I_{33}-\theta_2'\cdot I_{23}\right)
\end{aligned}\right\}$$

I_{22}, I_{33}, I_{23}: Moments of inertia

Equilibrium of bending moments

$$\left.\begin{aligned}
Q_3 &= \frac{\partial M_2}{\partial x_1}=E\left(\theta_2''\cdot I_{22}-\theta_3''\cdot I_{23}\right)\\
Q_2 &= -\frac{\partial M_3}{\partial x_1}=E\left(\theta_2''\cdot I_{23}-\theta_3''\cdot I_{33}\right)
\end{aligned}\right\}\rightarrow$$

$$E\left(\theta_2''x_3-\theta_3''x_2\right)=\frac{\left(Q_3I_{33}-Q_2I_{23}\right)x_3+\left(Q_2I_{22}-Q_3I_{23}\right)x_2}{I_{22}I_{33}-I_{23}{}^2}$$

THEORY OF UNIFORM SHEAR – DISPLACEMENT FIELD

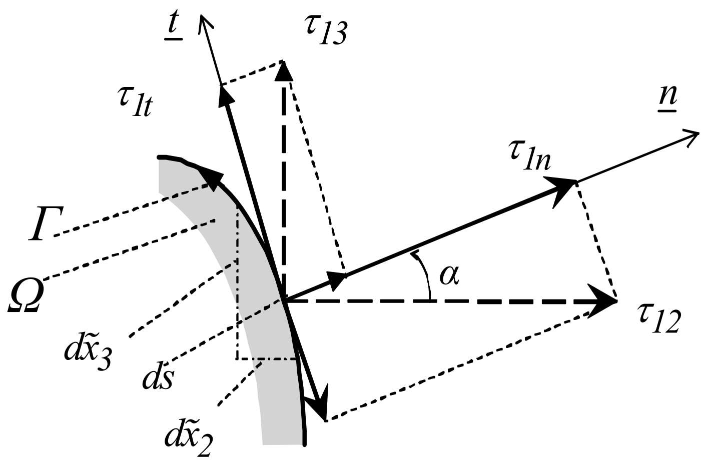
Poisson Partial Differential Equation

$$\nabla^2 \varphi_c(x_2, x_3) = \frac{\partial^2 \varphi_c}{\partial x_2^{\,2}} + \frac{\partial^2 \varphi_c}{\partial x_3^{\,2}} = f(x_2, x_3), \Omega$$

$$g(x_2, x_3) = \frac{1}{D}\left[(Q_3 I_{33} - Q_2 I_{23}) x_3 + (Q_2 I_{22} - Q_3 I_{23}) x_2\right] \qquad D = I_{22} I_{33} - I_{23}^{\,2}$$

$$f(x_2, x_3) = -\frac{1}{G} g(x_2, x_3)$$

Boundary Condition



$$\tau_{1n} = \tau_{12} n_2 + \tau_{13} n_3 = 0 \rightarrow$$

$$G\frac{\partial \varphi_c}{\partial n} = G\frac{\partial \varphi_c}{\partial x_2} n_2 + G\frac{\partial \varphi_c}{\partial x_3} n_3 = 0 \rightarrow$$

$$\frac{\partial \varphi_c}{\partial n} = 0 \;, \Gamma \quad \text{(Neumann)}$$

THEORY OF UNIFORM SHEAR – STRESS FIELD

Beam theory:

$\tau_{22} = \tau_{33} = \tau_{23} = 0$ & $\tau_{11} = -\left(\dfrac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2}\right) x_2 + \left(\dfrac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}^2}\right) x_3$

$Q_3 = \dfrac{\partial M_2}{\partial x_1}, Q_2 = -\dfrac{\partial M_3}{\partial x_1}$ (statically determinate beam

$\Rightarrow M_2, M_3$ are computed through the global equilibrium equations)

- **Analysis: Q_3** (Q_2 correspondingly and subsequent superposition of results)

Differential Equilibrium Equations of 3D Elasticity

(*Body forces neglected*)

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} = 0 \rightarrow \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} = \frac{Q_3}{I_{22} I_{33} - I_{23}^2}\left(x_2 I_{23} - x_3 I_{33}\right)$$

$$\left.\begin{aligned} \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} = 0 \rightarrow \frac{\partial \tau_{12}}{\partial x_1} = 0 \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} = 0 \rightarrow \frac{\partial \tau_{13}}{\partial x_1} = 0 \end{aligned}\right\} :$$

Shear stresses depend on x_2 & x_3, exclusively, that is they are the same at each cross section of the bar

Components of the infinitesimal strain tensor $(\nu \neq 0)$

$$\varepsilon_{11} = \frac{\tau_{11}}{E} \qquad \varepsilon_{22} = \varepsilon_{33} = -\frac{\nu}{E}\tau_{11} = -\nu\varepsilon_{11} \qquad \varepsilon_{23} = 0$$

$$\varepsilon_{12} = \frac{\tau_{12}}{2G} = \varepsilon_{12}(x_2, x_3) \qquad \varepsilon_{13} = \frac{\tau_{13}}{2G} = \varepsilon_{13}(x_2, x_3)$$

Compatibility Equations

Strain field → Satisfies 4 compatibility equations identically

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \tau_{13}}{\partial x_2} - \frac{\partial \tau_{12}}{\partial x_3}\right) = \frac{\nu Q_3 I_{33}}{(1+\nu)\left(I_{22}I_{33} - I_{23}^2\right)}$$

$$\frac{\partial}{\partial x_3}\left(\frac{\partial \tau_{13}}{\partial x_2} - \frac{\partial \tau_{12}}{\partial x_3}\right) = \frac{\nu Q_3 I_{23}}{(1+\nu)\left(I_{22}I_{33} - I_{23}^2\right)}$$

Stress Function

Shear stresses: → **They satisfy Compatibility Equations Identically**

$$\tau_{12} = \frac{Q_3}{B}\left(\frac{\partial \Phi}{\partial x_2} - d_2\right) \qquad \tau_{13} = \frac{Q_3}{B}\left(\frac{\partial \Phi}{\partial x_3} - d_3\right)$$

$\Phi(x_2, x_3)$: Stress function with continuous partial derivatives up to 2nd order

$$\mathbf{d} = d_2\mathbf{i_2} + d_3\mathbf{i_3} =$$

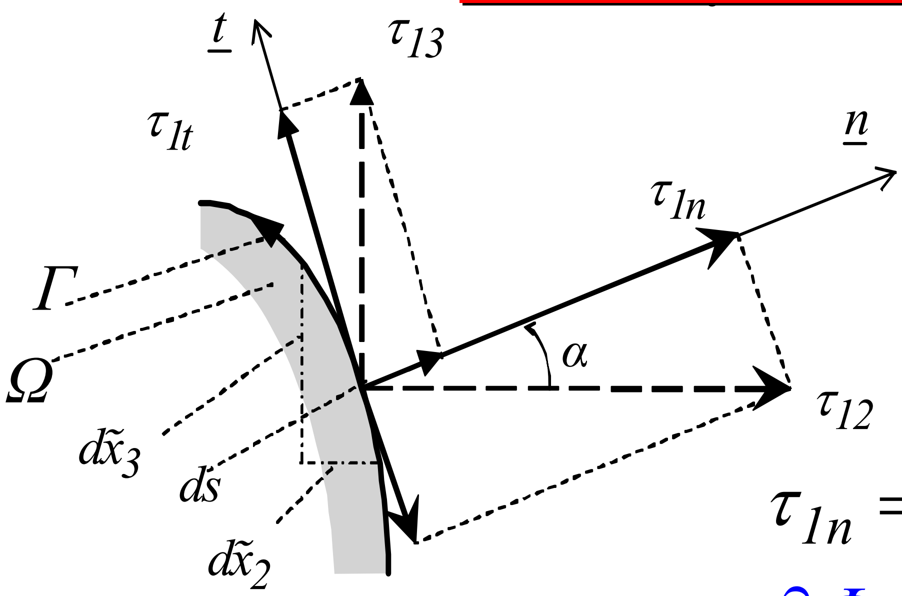
$$\left[\nu\left(I_{33}x_2x_3 - I_{23}\frac{x_2^{\,2} - x_3^{\,2}}{2}\right)\right]\mathbf{i_2} + \left[-\nu\left(I_{33}\frac{x_2^{\,2} - x_3^{\,2}}{2} + I_{23}x_2x_3\right)\right]\mathbf{i_3}$$

$$B = 2(1+\nu)\left(I_{22}I_{33} - I_{23}^2\right)$$

Poisson Partial Differential Equation

$$\nabla^2\Phi(x_2,x_3)=\frac{\partial^2\Phi}{\partial x_2{}^2}+\frac{\partial^2\Phi}{\partial x_3{}^2}=2(x_2I_{23}-x_3I_{33}),\Omega$$

Boundary Condition



$$\tau_{1n}=\tau_{12}n_2+\tau_{13}n_3=0\rightarrow$$

$$\frac{\partial\Phi}{\partial n}=d_2n_2+d_3n_3=\mathrm{n}\cdot\mathrm{d}\,,\Gamma$$

(Neumann)

Poisson Partial Differential Equation

$$\nabla^2\Theta = 2\left(I_{23}x_3 - I_{22}x_2\right), \Omega$$

Boundary Condition

$$\frac{\partial\Theta}{\partial n} = e_2 n_2 + e_3 n_3 = \mathrm{n}\cdot\mathrm{e}\ , \Gamma \qquad \text{(Neumann)}$$

$$\mathbf{e} = e_2\mathbf{i_2} + e_3\mathbf{i_3} =$$

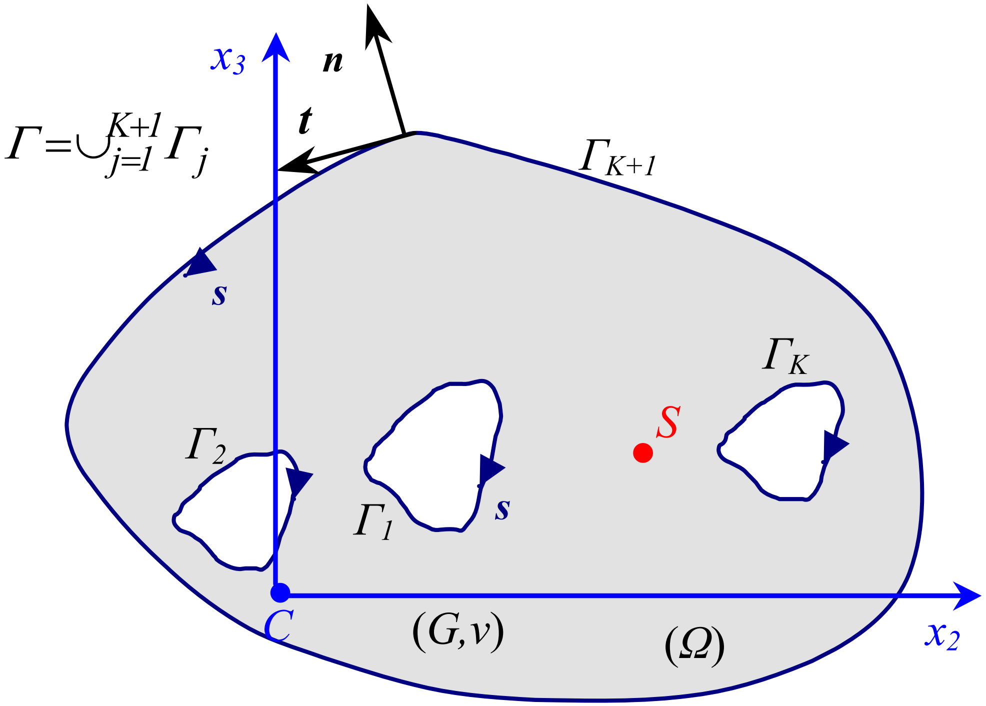
$$\left[\nu\left(I_{22}\frac{x_2{}^2 - x_3{}^2}{2} - I_{23}x_2x_3\right)\right]\mathbf{i_2} + \left[\nu\left(I_{23}\frac{x_2{}^2 - x_3{}^2}{2} + I_{22}x_2x_3\right)\right]\mathbf{i_3}$$

THEORY OF UNIFORM SHEAR

Shear Center (*S*)

- Point where Internal Shear Stress Resultant is subjected
- Poisson ratio $v = 0$ → **S.C. (*S*)** coincides with **C. of T. (*M*)** (Weber, 1924), (Trefftz, 1935)
- Determination: With respect to an arbitrary point $\boldsymbol{M_t^{ext} = M_t^{int}}$

M_t^{ext} : Twisting Moment at S.C. arising from externally applied forces
M_t^{int} : Twisting Moment arising from shear stresses due to transverse shear



$$\overbrace{Q_3 \cdot x_2^S - Q_2 \cdot x_3^S}^{Ext} = \overbrace{\int_\Omega \left(\tau_{13} x_2 - \tau_{12} x_3\right) d\Omega}^{Int} \rightarrow$$

THEORY OF UNIFORM SHEAR

SHEAR CENTER – DISPLACEMENT FIELD

- $Q_2 = 0 \,\&\, Q_3 = 1: \; x_2^S = G\int_\Omega \left(x_2 \dfrac{\partial \varphi_{cx_2}}{\partial x_3} - x_3 \dfrac{\partial \varphi_{cx_2}}{\partial x_2} \right) d\Omega$

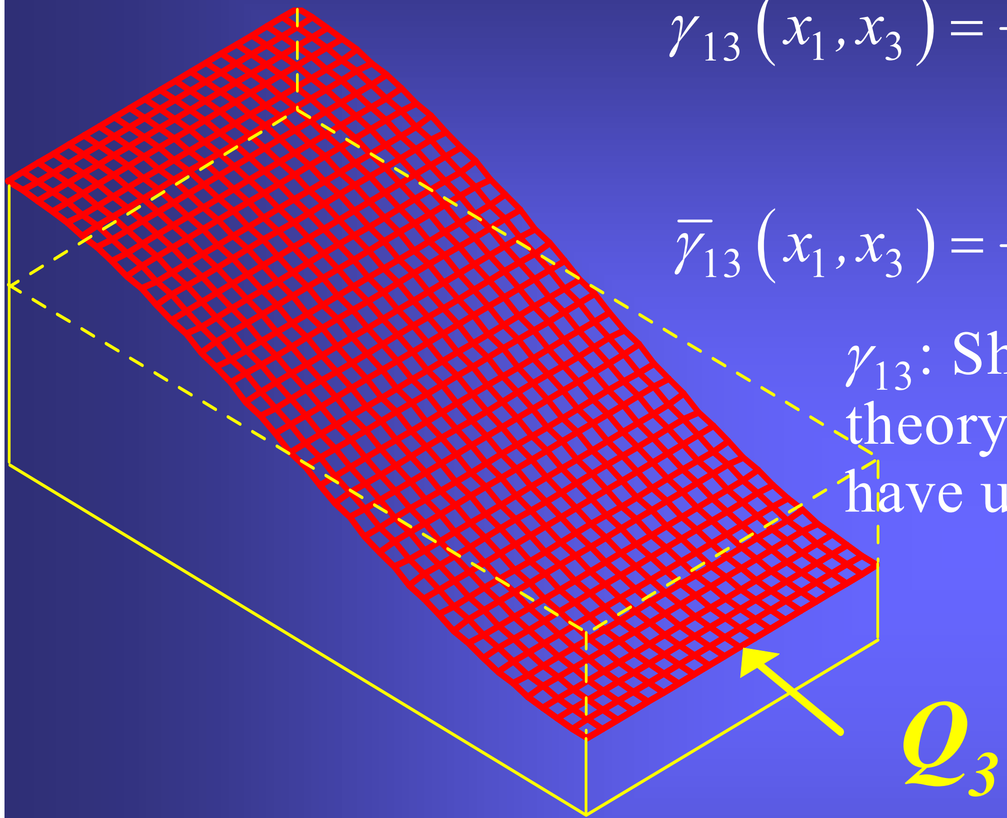
- $Q_2 = 1 \,\&\, Q_3 = 0: \; x_3^S = -G\int_\Omega \left(x_2 \dfrac{\partial \varphi_{cx_3}}{\partial x_3} - x_3 \dfrac{\partial \varphi_{cx_3}}{\partial x_2} \right) d\Omega$

SHEAR CENTER – STRESS FIELD

- $Q_2 = 0 \,\&\, Q_3 = 1: \; x_2^S = \dfrac{1}{B} \int_\Omega \left[x_2 \dfrac{\partial \Phi}{\partial x_3} - x_3 \dfrac{\partial \Phi}{\partial x_2} - x_2 d_3 + x_3 d_2 \right] d\Omega$

- $Q_2 = 1 \,\&\, Q_3 = 0: \; x_3^S = \dfrac{1}{B} \int_\Omega \left[x_3 \dfrac{\partial \Theta}{\partial x_2} - x_2 \dfrac{\partial \Theta}{\partial x_3} - x_3 e_2 + x_2 e_3 \right] d\Omega$

THEORY OF UNIFORM SHEAR



$$\tilde{\gamma}_{13}(x_1, x_3) = \frac{\tau_{13}(x_1, x_3)}{G}$$: Actual shear strains

$$\bar{\gamma}_{13}(x_1, x_3) = \frac{\int_{\Omega} \tilde{\gamma}_{13}(x_1, x_3)}{A}$$: Average shear strains

γ_{13}: Shear strains of Timoshenko beam theory $\Rightarrow$ They need correction since they have unsatisfactory (constant) distribution

Shear Deformation Coefficient α_3 (>1)

$$\gamma_{13} = a_3 \bar{\gamma}_{13}$$

Shear Correction Factor κ_3 (<1)

$$\bar{\gamma}_{13} = \frac{1}{a_3} \gamma_{13} = \kappa_3 \gamma_{13}$$

$$Q_3 = G\kappa_3 A\gamma_{13} = GA_{s3}\gamma_{13}$$

$$A_{s3} = \frac{1}{a_3} A = \kappa_3 A$$: **Effective Shear Area**

THEORIES OF SHEAR DEFORMATION COEFFICIENTS

1) Timoshenko Theory (1921, 1922): $\kappa_3 = \dfrac{\text{Average value of shear stresses}}{\text{Actual shear stress at centroid}}$ (if centroid does not lie in the cross section ?)

2) Cowper Theory (1966): Global equilibrium equations formulated by integrating the 3d elasticity differential equilibrium equations

3) Energy Approach (Bach & Baumann, 1924): The formulas of the approximate shear strain energy per unit length and the exact one are equated

α_3 must depend on the ratio of the sides (b/h)

$h\downarrow$ that is $b/h\uparrow$ then $\kappa_3 = 1/a_3 \rightarrow 0$ so that $\tilde{\gamma}_{13} = \dfrac{1}{a_3}\gamma_{13} = \kappa_3\gamma_{13} \rightarrow 0$

If α3 is independent of b/h then $\dfrac{GA_{s3}}{EI_{22}} \rightarrow$ Unacceptably large values !!! $\begin{cases} \text{Unrealistic results} \\ \text{FEM: “Shear–Locking”} \end{cases}$

Shear Deformation Coefficients

Exact formula of shear strain energy per unit length =
Approximate formula of shear strain energy per unit length

$$\overbrace{\int_{\Omega} \frac{\tau_{12}^2 + \tau_{13}^2}{2G} d\Omega}^{U_{exact}} = \overbrace{\frac{\alpha_2 Q_2^2}{2AG} + \frac{\alpha_3 Q_3^2}{2AG} + \frac{\alpha_{23} Q_2 Q_3}{AG}}^{U_{appr}}$$

Principal Shear System

$$tan\, 2\varphi^S = \frac{2a_{23}}{a_2 - a_3}$$

$\neq$

Principal Bending System

$$tan\, 2\varphi^B = \frac{2I_{23}}{I_{22} - I_{33}}$$

⟶ **Bending Deflections: <u>COUPLED</u>**

Axis of symmetry → $\varphi^S = \varphi^B$

THEORY OF UNIFORM SHEAR

SHEAR DEFORMATION COEFFICIENTS
DISPLACEMENT FIELD

- $\{Q_2 \neq 0,\ Q_3 = 0\}:$

$$a_2 = \frac{AG^2}{Q_2^2} \iint_{\Omega} \left[\left(\frac{\partial \varphi_{c2}}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi_{c2}}{\partial x_3} \right)^2 \right] d\Omega$$

- $\{Q_2 = 0,\ Q_3 \neq 0\}:$

$$a_3 = \frac{AG^2}{Q_3^2} \iint_{\Omega} \left[\left(\frac{\partial \varphi_{c3}}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi_{c3}}{\partial x_3} \right)^2 \right] d\Omega$$

- $\{Q_2 \neq 0,\ Q_3 \neq 0\}:$

$$a_{23} = \frac{AG^2}{Q^2} \iint_{\Omega} \left[\left(\frac{\partial \varphi_{c23}}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi_{c23}}{\partial x_3} \right)^2 \right] d\Omega - \frac{AG^2}{Q^2} \iint_{\Omega} \left[\left(\frac{\partial \varphi_{c3}}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi_{c3}}{\partial x_3} \right)^2 \right] d\Omega$$

$$- \frac{AG^2}{Q^2} \iint_{\Omega} \left[\left(\frac{\partial \varphi_{c2}}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi_{c2}}{\partial x_3} \right)^2 \right] d\Omega$$

THEORY OF UNIFORM SHEAR

SHEAR DEFORMATION COEFFICIENTS

STRESS FIELD

- $\{Q_2 \neq 0,\ Q_3 = 0\}:$

$$\alpha_2 = \frac{A}{B^2} \int_{\Omega} (\nabla \boldsymbol{\Theta} - \mathbf{e}) \cdot (\nabla \boldsymbol{\Theta} - \mathbf{e})\, d\Omega$$

- $\{Q_2 = 0,\ Q_3 \neq 0\}:$

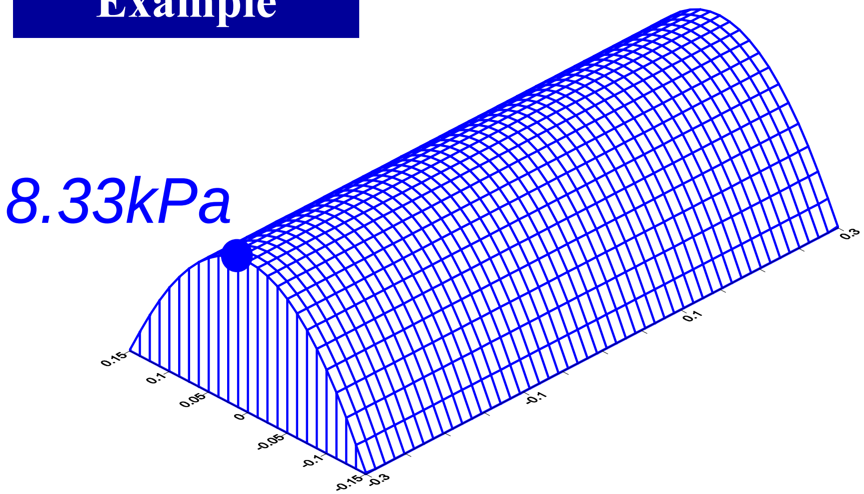
$$\alpha_3 = \frac{A}{B^2} \int_{\Omega} (\nabla \boldsymbol{\Phi} - \mathbf{d}) \cdot (\nabla \boldsymbol{\Phi} - \mathbf{d})\, d\Omega$$

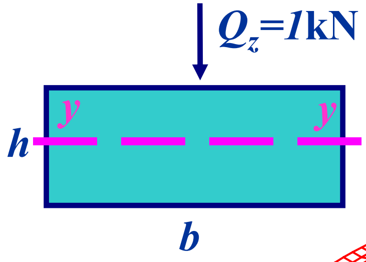
- $\{Q_2 \neq 0,\ Q_3 \neq 0\}:$

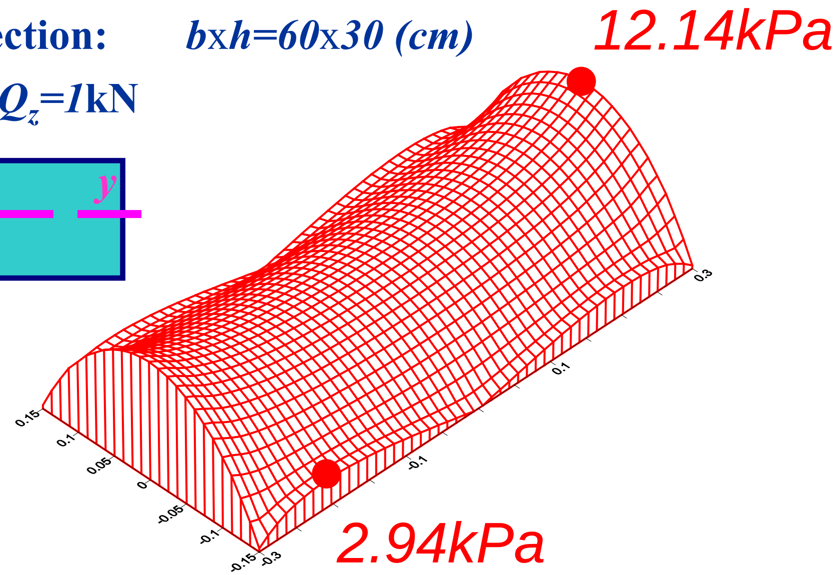
$$\alpha_{23} = 2\frac{A}{B^2} \int_{\Omega} \left[(\nabla \boldsymbol{\Phi} - \mathbf{d}) \cdot (\nabla \boldsymbol{\Theta} - \mathbf{e}) \right] d\Omega$$

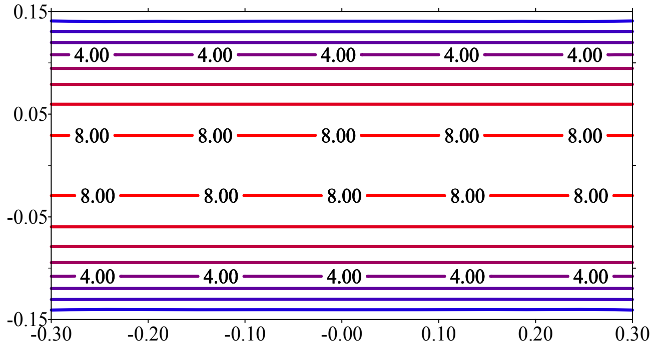
Example

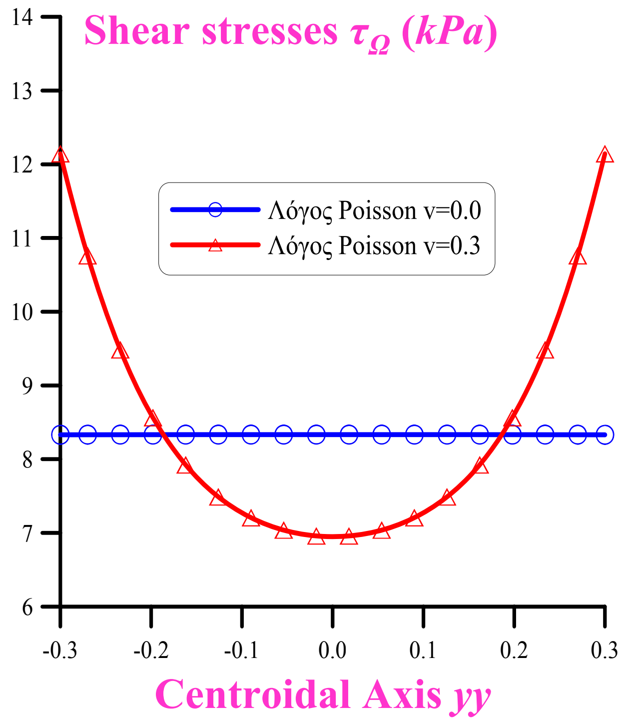
Rectangular section: ***b*x*h*=60x30 (cm)**

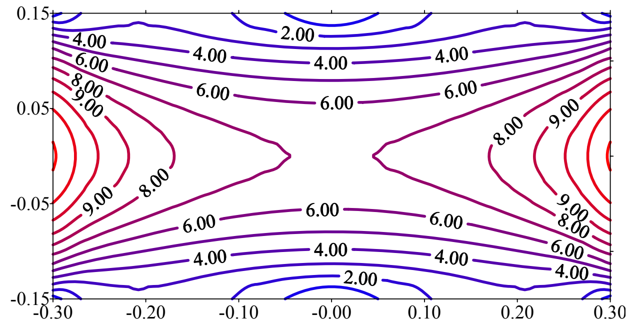












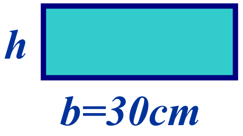
Poisson ratio:
v=0.0

Poisson ratio:
v=0.3

TBT: $\tau_{xz}^{max} = 1.5\dfrac{Q_z}{A} = 8.33kPa$

Discrepancy ≈ 31%

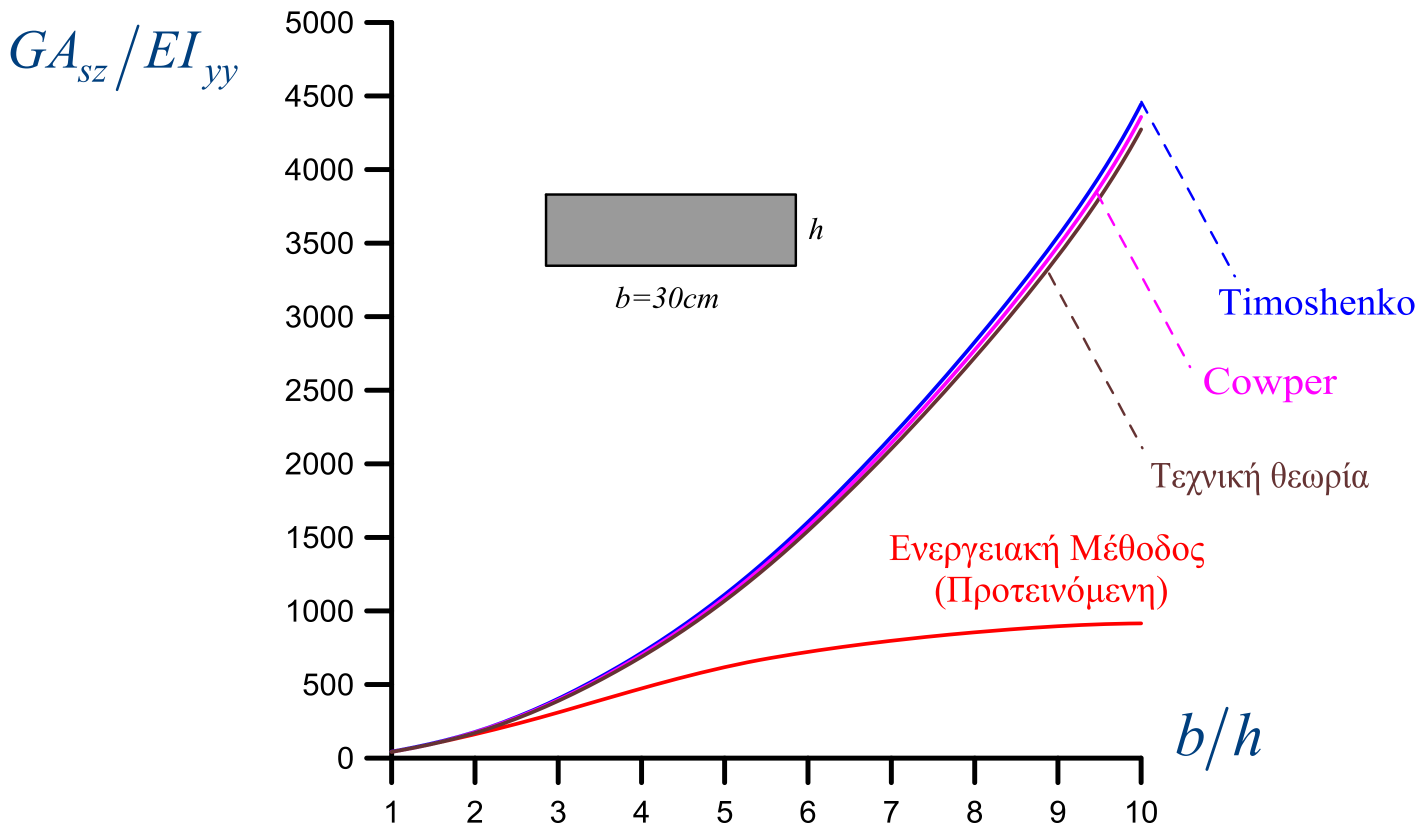
Example



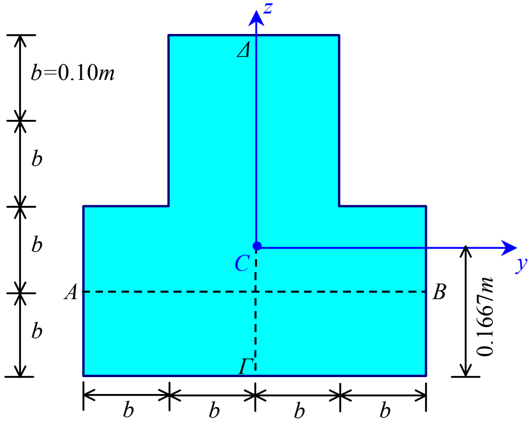
“Shear Locking” (Artificial, Spurious (Shear) Rigidity induced)

Λόγοι αντιστάσεων GA_{sz}/EI_{yy} $(1/m^2)$ με $v = 0.3$ και $b = 30cm$

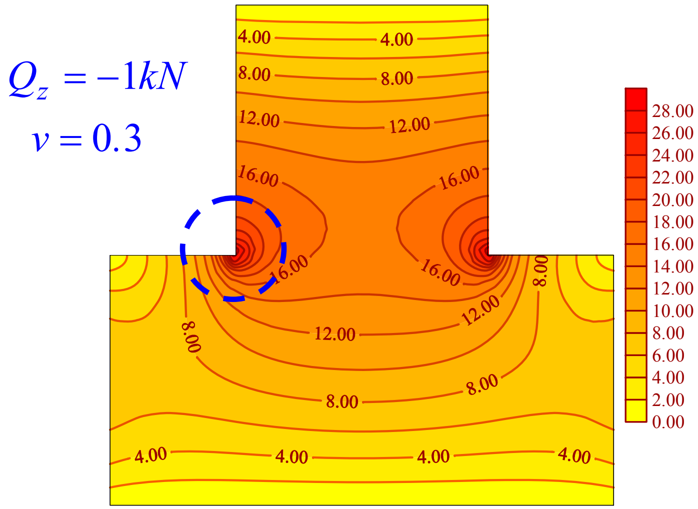
b/h	Ενεργειακή μέθοδος (προτεινόμενη)	Timoshenko	Cowper	Τεχνική θεωρία
1	42.47	44.45	43.57	42.74
2	160.93	177.78	174.29	170.94
5	617.77	1111.16	1089.35	1068.37
10	915.99	4444.62	4357.38	4273.51
50	958.61	111115.55	108934.59	106837.61
100	957.97	444462.22	435738.39	427350.43

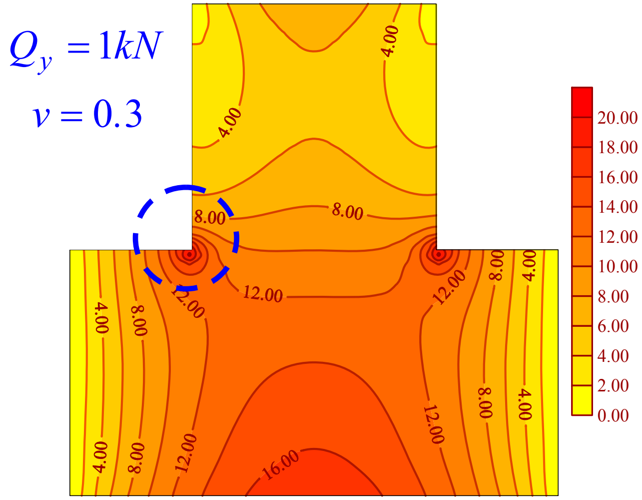


Example



T-shaped cross section

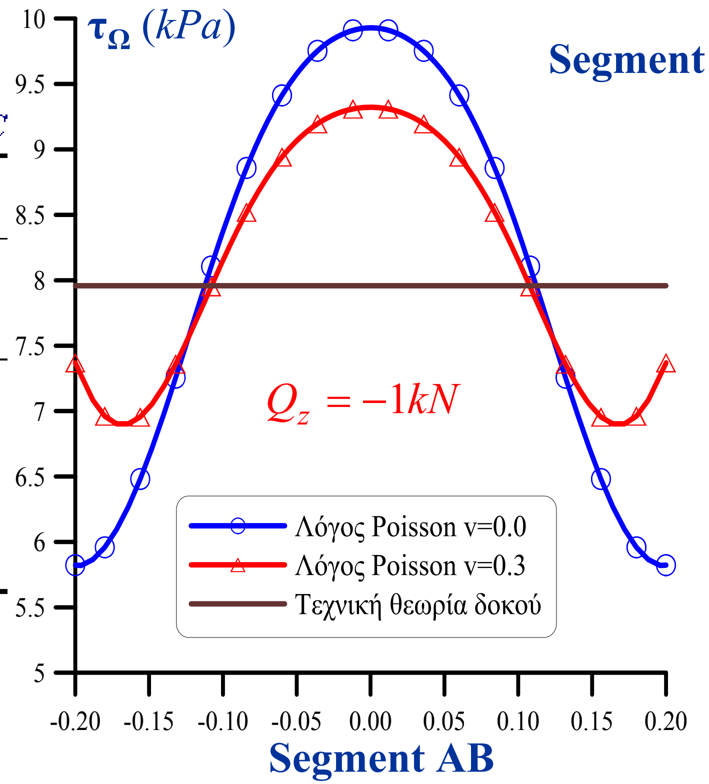


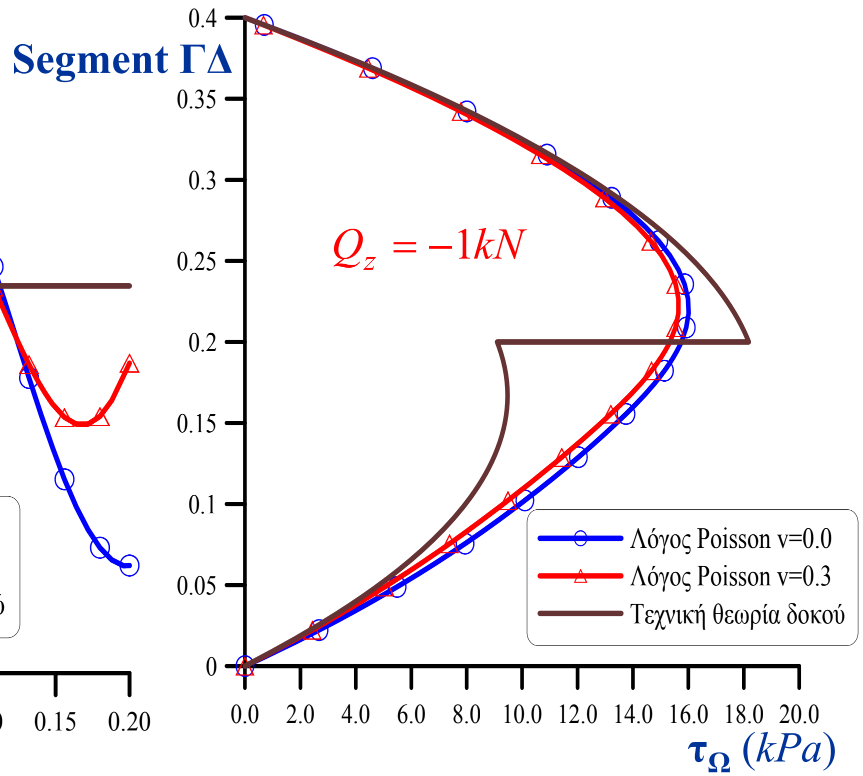


Κέντρο διάτμησης $z_S(m)$ ως προς C

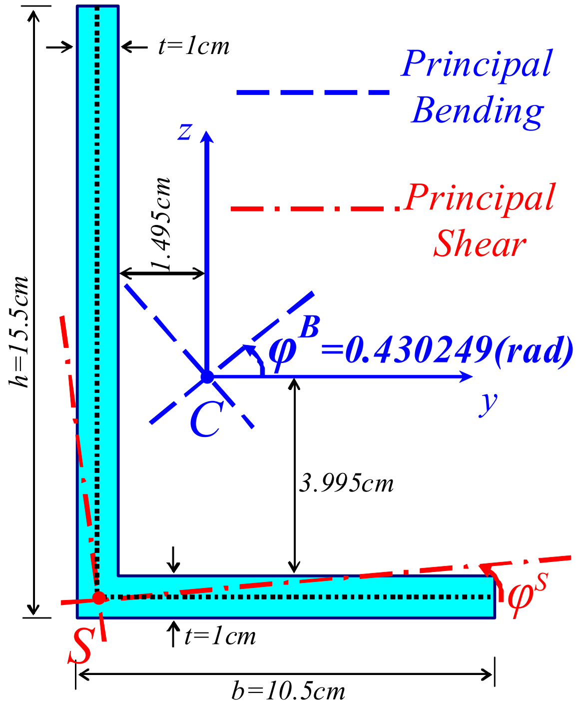
v	$z_S(m)$	
	BEM	FEM [Gruttmann 2001]
0.00	-0.0222	-0.0221
0.15	-0.0228	–
0.25	-0.0232	–
0.30	-0.0233	–
0.50	-0.0238	–

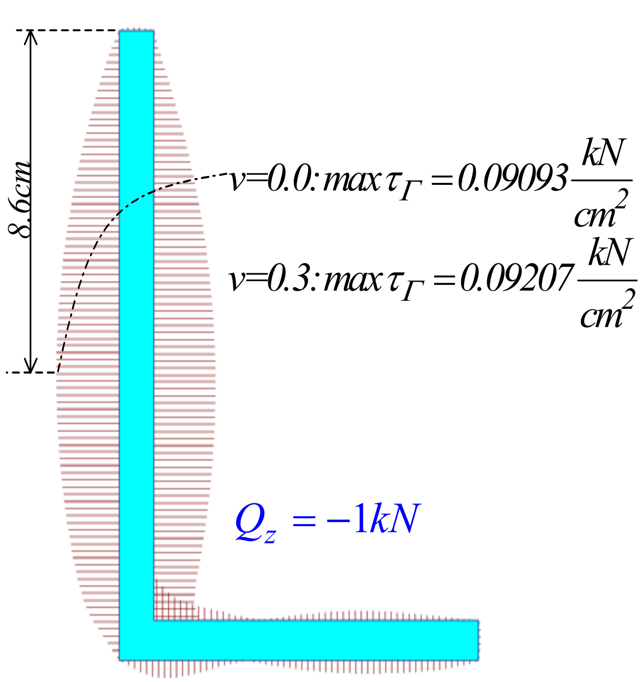
Poisson ratio ν → Does not affect shear center

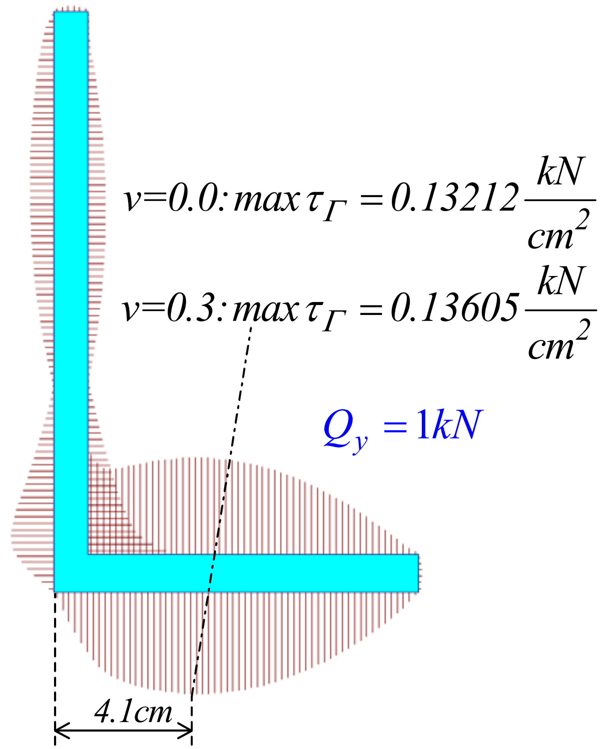




Example — Thin walled L-shaped cross section







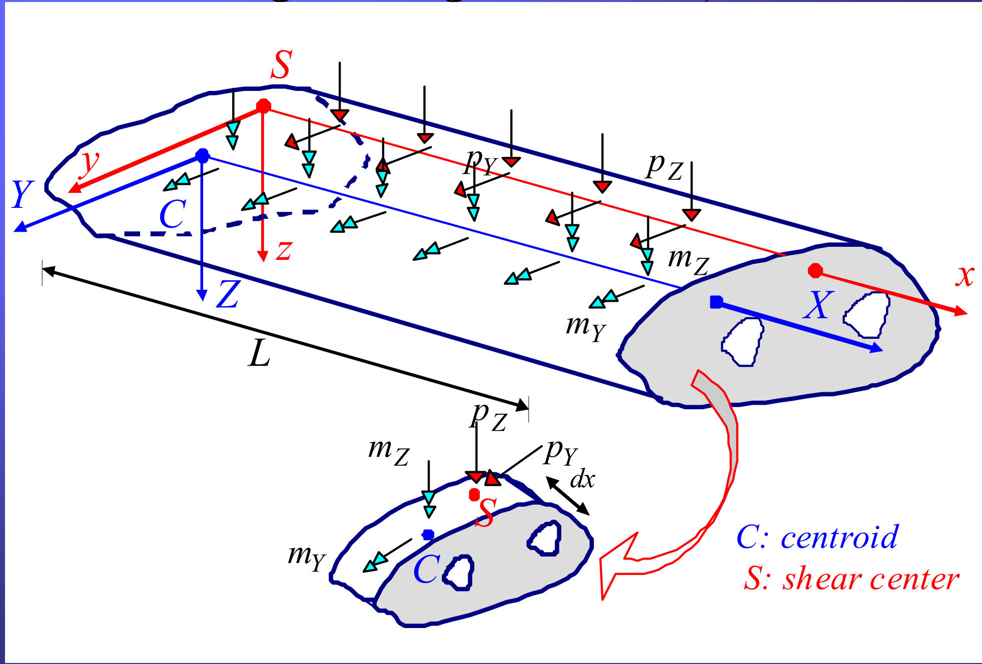
Συντεταγμένες κέντρου διάτμησης y_S, z_S (m) ως C

ν	y_S (m)		z_S (m)	
	BEM	FEM [Schramm 1997]	BEM	FEM [Schramm 1997]
0.00	-1.9983	-1.9976	-4.4254	-4.4239
0.20	-1.9982	–	-4.4256	–
0.30	-1.9982	-1.9976	-4.4257	-4.4243
0.40	-1.9982	–	-4.4257	–
	Θεωρία Λεπτότοιχων Ράβδων			
	-1.995		-4.495	

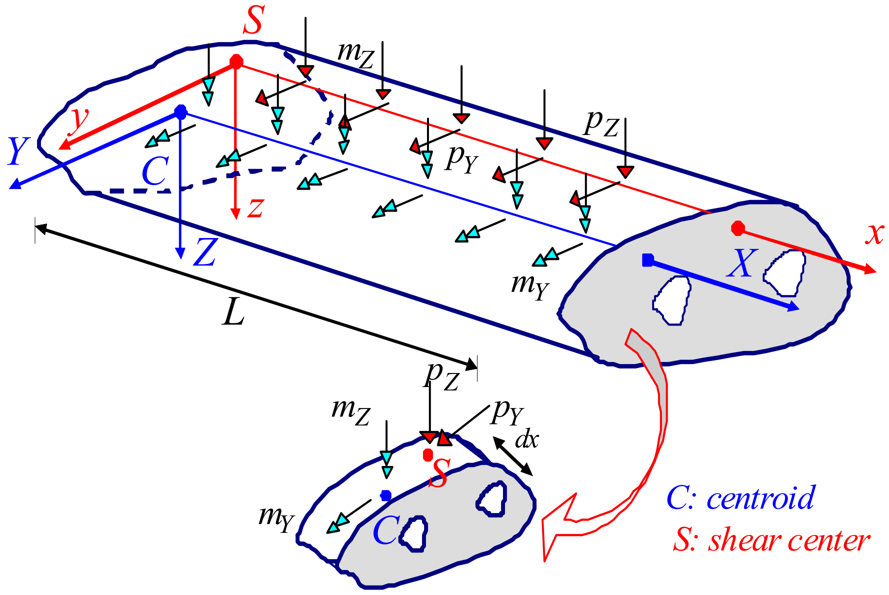
Small influence of Poisson ratio ν in stresses and Shear Center at thin walled cross section bars

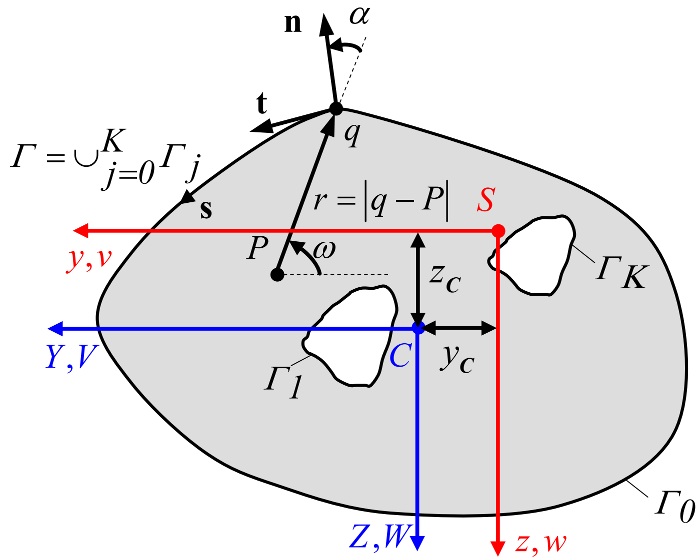
ASSUMPTIONS OF TIMOSHENKO BEAM THEORY

- The bar is straight.
- The bar is prismatic.
- Distortional deformations of the cross section are not allowed (γyz=0, distortion neglected).
- The material of the bar is homogeneous, isotropic, continuous (no cracking) and linearly elastic: Constitutive relations of linear elasticity are valid.
- External transverse forces pass through the cross section's shear center. Torsional and axial forces are not considered (torsionless bending loading conditions).
- Deflections and bending rotations are considered to be small (geometrically linear theory).
- **Cross sections remain plane after deformation**.
- The distribution of stresses at the bar ends is such so that all the aforementioned assumptions are valid.



TIMOSHENKO BEAM THEORY





Use of the principal **shear system** $CXYZ$ passing through the centroid C

Displacement Field: (Arising from the plane sections hypothesis)

$$\bar{u}(x,y,z) = \theta_Y(x)(z - z_C) - \theta_Z(x)(y - y_C) = \theta_Y(x)Z - \theta_Z(x)Y$$

$$\bar{v}(x,z) = v(x) \qquad\qquad \theta_Y(x) \neq -w'(x)$$

$$\bar{w}(x,y) = w(x) \qquad\qquad \theta_Z(x) \neq v'(x)$$

Components of the Infinitesimal Strain Tensor
(Geometrically linear theory)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{d\theta_Y}{dx} Z - \frac{d\theta_Z}{dx} Y$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{dv}{dx} - \theta_Z$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{dw}{dx} + \theta_Y$$

TIMOSHENKO BEAM THEORY

Components of the Cauchy Stress Tensor (ν=0)

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] = E\left(\frac{d\theta_Y}{dx}Z - \frac{d\theta_Z}{dx}Y\right)$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] = 0$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] = 0$$

$$\tau_{yz} = G \cdot \gamma_{yz} = 0$$

$$\tau_{xy} = G \cdot \gamma_{xy} = G\left(-\theta_Z + v'\right)$$

$$\tau_{xz} = G \cdot \gamma_{xz} = G\left(\theta_Y + w'\right)$$

Differential Equilibrium Equations of 3D Elasticity

(*Body forces neglected*)

Not Satisfied! →
Inconsistency of Timoshenko Beam Theory:

Overall equilibrium of the bar is satisfied (energy principle). The violation of the longitudinal equilibrium equation (along x) and of the associated boundary condition is due to the unsatisfactory distribution of the shear stresses arising from the plane sections hypothesis. Thus, in order to correct **at the global level** this unsatisfactory distribution of shear stresses, **we introduce shear correction factors in the cross sectional shear rigidities at the global equilibrium equations**

$$\tau_{xy} = G \cdot \gamma_{xy} = G\left(-\theta_Z + v'\right) \qquad \tau_{xz} = G \cdot \gamma_{xz} = G\left(\theta_Y + w'\right)$$

constant distribution: unsatisfactory

TIMOSHENKO BEAM THEORY

Stress Resultants

- **Shear stress resultants:** $Q_y = \int_{\Omega} \tau_{xy} d\Omega = GA_y \left(\frac{dv}{dx} - \theta_Z \right)$

$$Q_z = \int_{\Omega} \tau_{xz} d\Omega = GA_z \left(\theta_Y + \frac{dw}{dx} \right)$$

A_y, A_z: Shear areas with respect to the y,z axes

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad A_z = \kappa_z A = \frac{1}{a_z} A$$

κ_y, κ_z: shear correction factors (<1)

a_y, a_z: shear deformation coefficients(>1)

(From the assumed displacement field we would have obtained shear rigidities GA which are larger than the actual ones)

Since we are working with the principal shear system of axes $\Rightarrow$ $a_{yz} = 0$

Thus the relations of shear stress resultants with respect to the kinematical components are decoupled

TIMOSHENKO BEAM THEORY

Stress Resultants

In general we would have $a_{yz} \neq 0$:

$$Q_y = \int_{\Omega} \tau_{xy} d\Omega = G A_y \left(\frac{dv}{dx} - \theta_Z \right)$$

$$Q_z = \int_{\Omega} \tau_{xz} d\Omega = G A_z \left(\theta_Y + \frac{dw}{dx} \right)$$

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad A_z = \kappa_z A = \frac{1}{a_z} A \qquad \left(A_{yz} = \kappa_{yz} A = \frac{1}{a_{yz}} A \right)$$

U_{exact} → From uniform shear beam theory

U_{appr} → From this theory

$$\overbrace{\int_{\Omega} \frac{\tau_{xy}^2 + \tau_{xz}^2}{2G} d\Omega}^{U_{exact}} = \overbrace{\frac{\alpha_y Q_y^{\,2}}{2AG} + \frac{\alpha_z Q_z^{\,2}}{2AG} + \frac{\alpha_{yz} Q_y Q_z}{AG}}^{U_{appr}}$$

TIMOSHENKO BEAM THEORY

Stress Resultants

- **Bending moments:** $M_Y = \int_{\Omega} \sigma_{xx} Z d\Omega \qquad M_Z = -\int_{\Omega} \sigma_{xx} Y d\Omega$

Bending moments are defined with respect to the principal shear system of axes passing through the centroid of the cross section

$$M_Y = \int_{\Omega} \sigma_{xx} Z d\Omega = \int_{\Omega} EZ^2 \frac{d\theta_Y}{dx} d\Omega - \int_{\Omega} EYZ \frac{d\theta_Z}{dx} d\Omega = EI_Y \frac{d\theta_Y}{dx} - EI_{YZ} \frac{d\theta_Z}{dx}$$

$$M_Z = -\int_{\Omega} \sigma_{xx} Y d\Omega = \int_{\Omega} EY^2 \frac{d\theta_Z}{dx} d\Omega - \int_{\Omega} EYZ \frac{d\theta_Y}{dx} d\Omega = EI_Z \frac{d\theta_Z}{dx} - EI_{YZ} \frac{d\theta_Y}{dx}$$

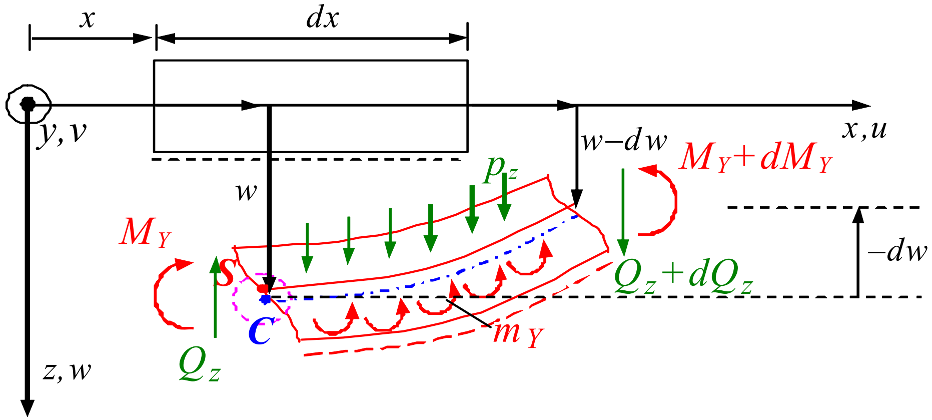
$$I_Y = \int_{\Omega} Z^2 d\Omega, \; I_Z = \int_{\Omega} Y^2 d\Omega, \; I_{YZ} = \int_{\Omega} YZ d\Omega:$$

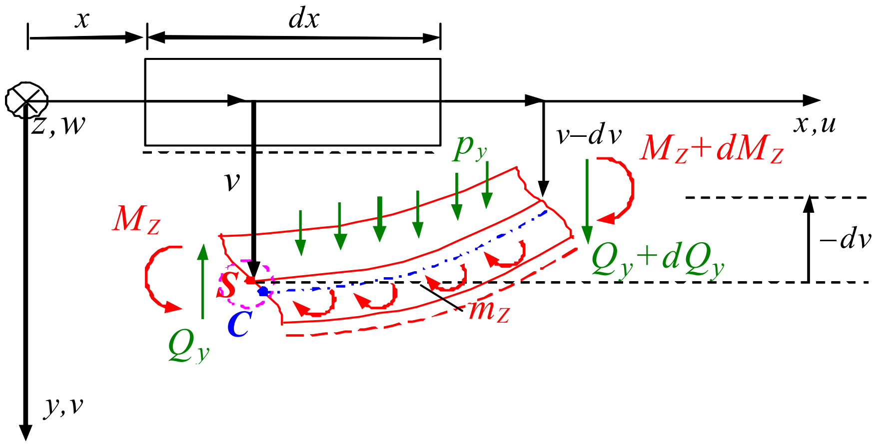
Moments of inertia with respect to the centroid of the cross section

TIMOSHENKO BEAM THEORY

Global Equilibrium Equations & Boundary conditions

Method of Equilibrium or Energy Method





TOTAL POTENTIAL ENERGY

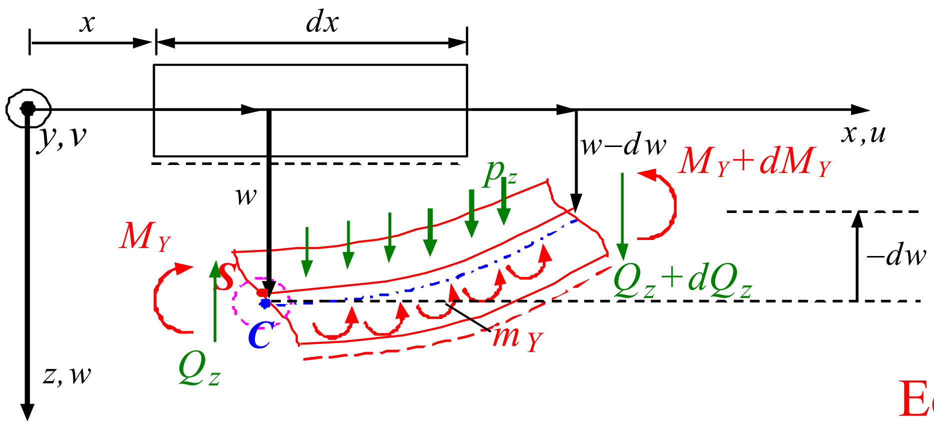
$$\frac{\partial F}{\partial \theta_1} - \frac{d}{d\tilde{x}_1}\frac{\partial F}{\partial \theta_1'} + \frac{d^2}{d\tilde{x}_1^2}\frac{\partial F}{\partial \theta_1''} = 0$$

(*Euler–Lagrange eqns*)

TIMOSHENKO BEAM THEORY

Global Equilibrium Equations & Boundary conditions

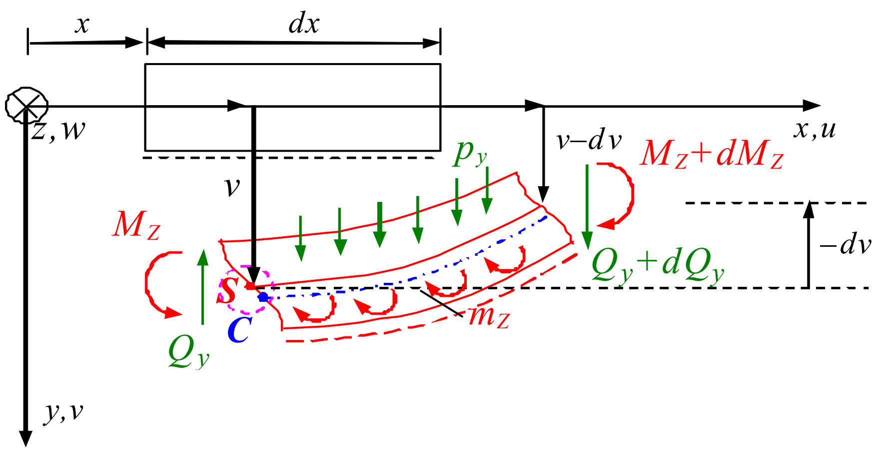
Method of Equilibrium



Equilibrium of bending moments

$$\frac{dM_Y}{dx}-Q_z+m_Y=0$$

$$\frac{dM_Z}{dx}+Q_y+m_Z=0$$



Equilibrium of transverse shear forces

$$\frac{dQ_y}{dx}+p_Y=0$$

$$\frac{dQ_z}{dx}+p_Z=0$$

TIMOSHENKO BEAM THEORY

Global Equilibrium Equations & Boundary conditions

Equilibrium of bending moments

$$\frac{dM_Y}{dx} - Q_z + m_Y = 0 \Rightarrow EI_Y \frac{d^2\theta_Y}{dx^2} - EI_{YZ} \frac{d^2\theta_Z}{dx^2} - \frac{GA}{a_Z}\left(\theta_Y + \frac{dw}{dx}\right) + m_Y = 0 \quad (1)$$

$$\frac{dM_Z}{dx} + Q_y + m_Z = 0 \Rightarrow EI_Z \frac{d^2\theta_Z}{dx^2} - EI_{YZ} \frac{d^2\theta_Y}{dx^2} + \frac{GA}{a_Y}\left(\frac{dv}{dx} - \theta_Z\right) + m_Z = 0 \quad (2)$$

Equilibrium of transverse shear forces

$$\frac{dQ_y}{dx} + p_y = 0 \Rightarrow \frac{GA}{a_Y}\left(\frac{d^2v}{dx^2} - \frac{d\theta_Z}{dx}\right) + p_y = 0 \quad (3)$$

$$\frac{dQ_z}{dx} + p_z = 0 \Rightarrow \frac{GA}{a_Z}\left(\frac{d\theta_Y}{dx} + \frac{d^2w}{dx^2}\right) + p_z = 0 \quad (4)$$

Inside the bar interval

$$\beta_1 v + \beta_2 Q_y = \beta_3 \qquad \gamma_1 w + \gamma_2 Q_z = \gamma_3$$

$$\bar{\beta}_1 \theta_Z + \bar{\beta}_2 M_Z = \bar{\beta}_3 \qquad \bar{\gamma}_1 \theta_Y + \bar{\gamma}_2 M_Y = \bar{\gamma}_3$$

At the bar ends

→ Coupled system of equations due to principal shear system of axes and due to shear deformation effects

TIMOSHENKO BEAM THEORY

Global Equilibrium Equations & Boundary conditions

Combination of equations may be performed in order to uncouple the problem unknowns - Solution with respect to bending rotations (or deflections)

Resolution of rotations:

$$EI_Y \frac{d^3\theta_Y}{dx^3} - EI_{YZ} \frac{d^3\theta_Z}{dx^3} + \frac{dm_Y}{dx} + p_y = 0 \; (1'),\; (1) \text{ into } (3) \qquad \bar{\beta}_1\theta_Z + \bar{\beta}_2 M_Z = \bar{\beta}_3$$

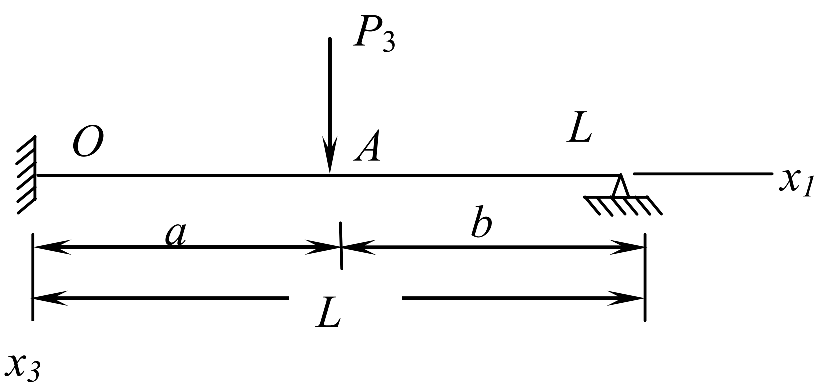
$$EI_Z \frac{d^3\theta_Z}{dx^2} - EI_{YZ} \frac{d^3\theta_Y}{dx^2} + \frac{dm_Z}{dx} - p_z = 0 \; (2'),\; (2) \text{ into } (4) \qquad \bar{\gamma}_1\theta_Y + \bar{\gamma}_2 M_Y = \bar{\gamma}_3$$

Resolution of deflections:

$$EI_Y \frac{d^2\theta_Y}{dx^2} - EI_{YZ} \frac{d^2\theta_Z}{dx^2} - \frac{GA}{a_Z}\left(\theta_Y + \frac{dw}{dx}\right) + m_Y = 0 \; (1) \qquad \gamma_1 w + \gamma_2 Q_z = \gamma_3$$

$$EI_Z \frac{d^2\theta_Z}{dx^2} - EI_{YZ} \frac{d^2\theta_Y}{dx^2} + \frac{GA}{a_Y}\left(\frac{dv}{dx} - \theta_Z\right) + m_Z = 0 \; (2) \qquad \beta_1 v + \beta_2 Q_y = \beta_3$$

TIMOSHENKO BEAM THEORY - EXAMPLE



→ Torsionless bending on the Ox1x3 plane

→ Find the elastic curve and the reactions of the beam using Timoshenko beam theory

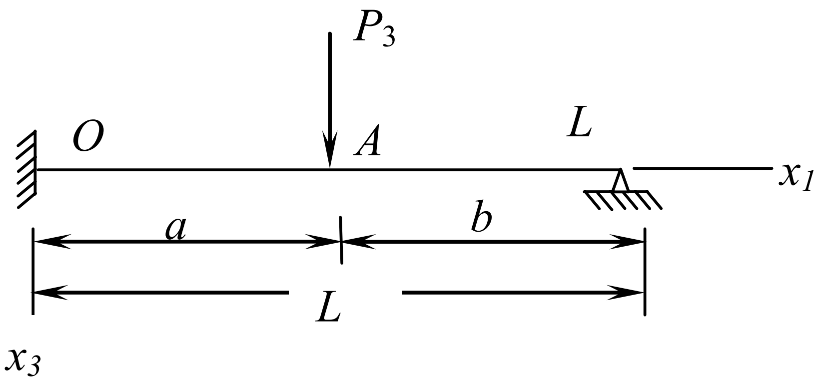
STEP 1: Equation of equilibrium $(1')$ (forces):

$$EI_2 \frac{d^3\theta_2}{dx_1^3} - EI_{23}\frac{d^3\theta_3}{dx_1^3} + \frac{dm_2}{dx_1} + p_2 = 0 \Rightarrow EI_2 \frac{d^3\theta_2}{dx_1^3} = 0 \Rightarrow$$

$$\frac{d^2}{dx_1^2}\left(EI_2 \frac{d\theta_2^{(1)}}{dx_1}\right) = 0,\ 0 < x_1 < a \qquad \text{(a)}$$

$$\frac{d^2}{dx_1^2}\left(EI_2 \frac{d\theta_2^{(2)}}{dx_1}\right) = 0,\ a < x_1 < L \qquad \text{(b)}$$

TIMOSHENKO BEAM THEORY - EXAMPLE



→ Torsionless bending to the Ox1x3 plane

→ Find the elastic curve and the reactions of the beam using Timoshenko beam theory

Integrate three times eqns (a,b):

$0 < x_1 < a$ $\qquad\qquad$ $a < x_1 < L$

$$Q_3^{(1)} = \frac{d}{dx_1}\left(EI_2 \frac{d\theta_2^{(1)}}{dx_1}\right) = C_1 \qquad Q_3^{(2)} = \frac{d}{dx_1}\left(EI_2 \frac{d\theta_2^{(2)}}{dx_1}\right) = C_2 \qquad \text{(c)}$$

$$M_2^{(1)} = EI_2 \frac{d\theta_2^{(1)}}{dx_1} = C_1 x_1 + C_3 \qquad M_2^{(2)} = EI_2 \frac{d\theta_2^{(2)}}{dx_1} = C_2 x_1 + C_4 \qquad \text{(d)}$$

$$EI_2 \theta_2^{(1)} = C_1 \frac{x_1^2}{2} + C_3 x_1 + C_5 \qquad EI_2 \theta_2^{(2)} = C_2 \frac{x_1^2}{2} + C_4 x_1 + C_6 \qquad \text{(e)}$$

TIMOSHENKO BEAM THEORY - EXAMPLE

STEP 2: Resolve constants C_1-C_6 by exploiting the boundary conditions (rotations, moments):

1. $\theta_2^{(1)}(x1)$ at $x_1 = 0$: $\theta_2^{(1)}(0) = 0 \xrightarrow{\text{relations (e)}} C_5 = 0$ (f)

2. $M_2^{(2)}(x1)$ at $x_1 = L$: $M_2^{(2)}(L) = 0 \xrightarrow{\text{relations (d)}} C_2 L + C_4 C_5 = 0$ (g)

3. Rotational continuity condition at $x_1 = a$: $\theta_2^{(1)}(a) = \theta_2^{(2)}(a)$ (h)

$$\xrightarrow{\text{relations (e)}} C_1 \frac{a^2}{2} + C_3 a = \frac{C_2 a^2}{2} + C_4 a + C_6 \quad \text{(i)}$$

4. Equilibrium conditions (forces, moments) at $x_1 = a$:

$$Q_3^{(1)}\left(a^-\right) = Q_3^{(2)}\left(a^+\right) + P_3 \quad \text{(j)}$$

$$M_2^{(1)}(a) = M_2^{(2)}(a) \quad \text{(k)}$$

$$\xrightarrow{\text{relations (c,d)}} C_1 = C_2 + P_3 \quad \text{(l)}$$

$$C_1 a + C_3 = C_2 a + C_4 \quad \text{(m)}$$

From relations (g), (i), (l), (m) $\Rightarrow$

$$C_1 = C_2 + P_3 \qquad C_3 = -C_2 L - aP_3 \qquad C_4 = C_2 L \qquad C_6 = -a^2 P_3 / 2 \quad \text{(n)}$$

TIMOSHENKO BEAM THEORY - EXAMPLE

STEP 3: Resolve deflections from equation of equilibrium (1) (moments):

$$EI_2\frac{d^2\theta_2}{dx_1^2}-EI_{23}\frac{d^2\theta_3}{dx_1^2}-\frac{GA}{a_3}\left(\theta_2+\frac{du_3}{dx_1}\right)+m_2=0\Rightarrow EI_2\frac{d^2\theta_2}{dx_1^2}-k_3GA\left(\theta_2+\frac{du_3}{dx_1}\right)=0\Rightarrow$$

$$k_3GA\left(\theta_2^{(1)}+\frac{du_3^{(1)}}{dx_1}\right)=Q_3^{(1)}(x_1)=\frac{d}{dx_1}\left(EI_2\frac{d\theta_2^{(1)}}{dx_1}\right),\; 0<x_1<a \qquad \text{(o)}$$

$$k_3GA\left(\theta_2^{(2)}+\frac{du_3^{(2)}}{dx_1}\right)=Q_3^{(2)}(x_1)=\frac{d}{dx_1}\left(EI_2\frac{d\theta_2^{(2)}}{dx_1}\right),\; a<x_1<L \qquad \text{(o)}$$

Substitute the values of constants (f), (n) into relations (e) and (c) and the resulting expressions in (o), we get:

$$\frac{du_3^{(1)}}{dx_1}=\frac{C_2+P_3}{k_3GA}-\frac{(C_2+P_3)x_1^2}{2EI_2}+\frac{(C_2L+aP_3)x_1}{EI_2} \qquad \text{(p)}$$

$$\frac{du_3^{(2)}}{dx_1}=\frac{C_2}{k_3GA}-\frac{C_2x_1^2}{2EI_2}+\frac{C_2Lx_1}{EI_2}+\frac{a^2P_3}{2EI_2} \qquad \text{(p)}$$

TIMOSHENKO BEAM THEORY - EXAMPLE

Integrate once relations (p):

$$u_3^{(1)}(x_1) = \frac{(C_2 + P_3)x_1}{k_3 GA} - \frac{(C_2 + P_3)x_1^3}{6EI_2} + \frac{(C_2 L + aP_3)x_1^2}{2EI_2} + C_7 \quad \text{(q)}$$

$$u_3^{(2)}(x_1) = \frac{C_2 x_1}{k_3 GA} - \frac{C_2 x_1^3}{6EI_2} + \frac{C_2 L x_1^2}{2EI_2} + \frac{a^2 P_3 x_1}{2EI_2} + C_8 \quad \text{(q)}$$

STEP 4: Resolve constants C_2, C_7, C_8 by exploiting the boundary conditions (translations):

$$1.\ u_3^{(1)}(x1) \text{ at } x_1 = 0 :\ u_3^{(1)}(0) = 0 \xrightarrow{\text{relations (q)}} C_7 = 0 \quad \text{(r)}$$

$$2.\ u_3^{(2)}(x1) \text{ at } x_1 = L :\ u_3^{(2)}(L) = 0 \xrightarrow{\text{relations (q)}} \frac{C_2 L}{k_3 GA} + \frac{C_2 L^3}{3EI_2} + \frac{a^2 L P_3}{2EI_2} + C_8 = 0 \quad \text{(s)}$$

3. Translational continuity condition at $x_1 = a :\ u_3^{(1)}(a) = u_3^{(2)}(a)$

$$\xrightarrow{\text{relations (q)}} C_8 = \frac{P_3 a}{k_3 GA} - \frac{P_3 a^3}{6EI_2} \quad \text{(t)}$$

From relations (t), (s) $\Rightarrow C_2 = -\dfrac{P_3 a}{2L^3}\left(\dfrac{3aL - a^2 + 2kL^2}{1 + k}\right),\ k = \dfrac{3EI_2}{\lambda_3 GAL^2} \Rightarrow u_3(x1)\ldots$ (u)

TIMOSHENKO BEAM THEORY - EXAMPLE

→ For a rectangular cross section b×h and ν=1/3 we have $I_2 = \frac{bh^3}{12}$, $k = \frac{3h^2(1+\nu)}{4L^2}$

→ From relations (n),(u),(c),(d) we get the expressions of shear forces and bending moments:

$$Q_3^{(1)} = C_1 = C_2 + P_3 = -\frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] + P_3$$

$$Q_3^{(2)} = C_2 = -\frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right]$$

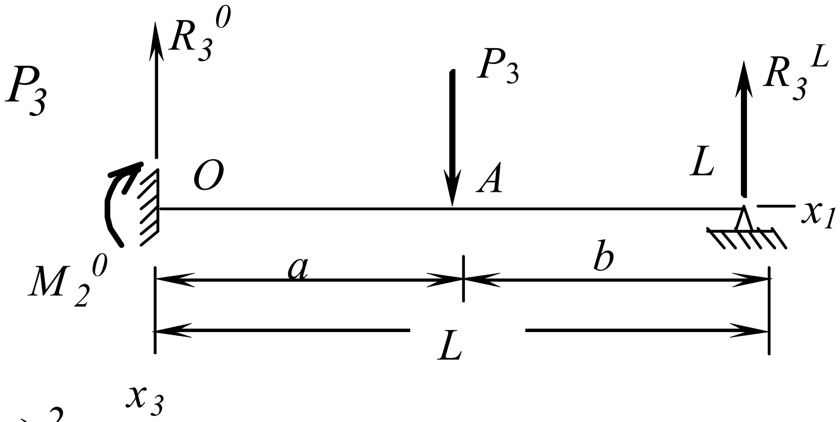
$$M_2^{(1)} = C_1 x_1 + C_3 = -C_2(L - x_1) - P_3(a - x_1)$$

$$M_2^{(2)} = C_2 x_1 + C_4 = -C_2(L - x_1)$$

TIMOSHENKO BEAM THEORY - EXAMPLE

→The reactions of the beam are:

$$R_3^{\,0} = Q_3^{(1)}(0) = -\frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] + P_3$$



$$R_3^{\,L} = -Q_3^{(2)}(L) = \frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right]$$

$$M_2^0 = M_2^{(1)}(0) = -C_2 L - P_3 a = \frac{P_3 a}{2(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] - P_3 a$$

$$M_2^L = M_2^{(2)}(L) = 0$$

→For small h/L ratios (h/L<10) which usually occur in practice, shear deformations influence negligibly internal actions and reactions of beams

Thank You