Ε.Ι. Σαπουντζάκης

Καθηγητής ΕΜΠ

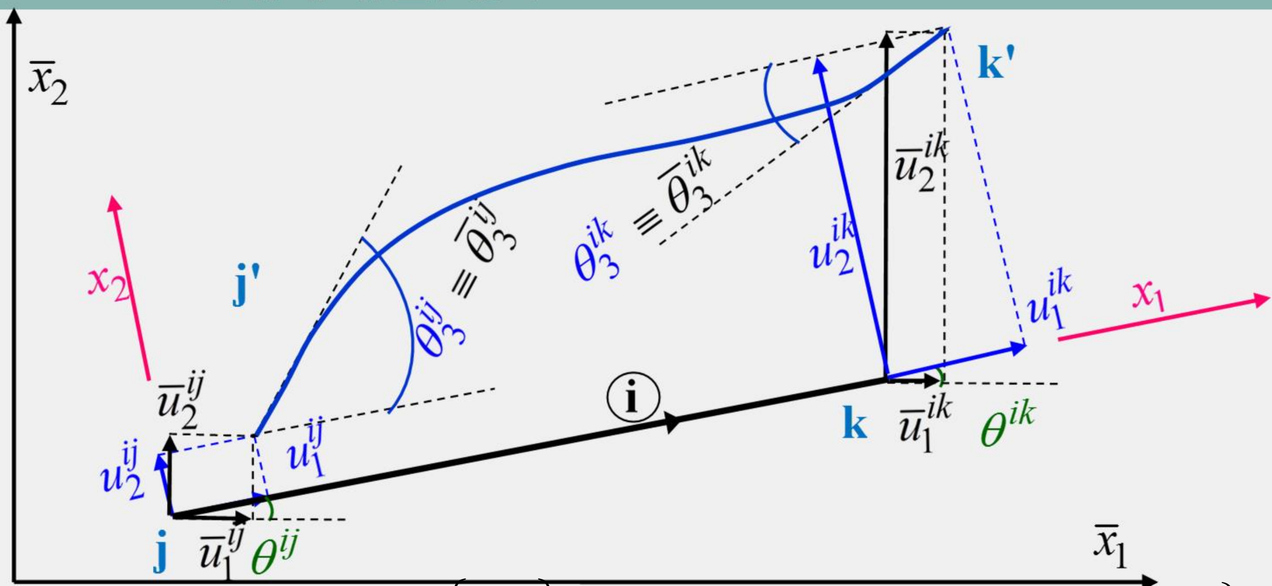
Μ. Νεραντζάκη

Αναπλ. Καθηγήτρια ΕΜΠ

[illegible]

[illegible]

[illegible]



$$\left\{D^{ij}\right\} = \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \end{Bmatrix}$$

[illegible] *j*

$$\left\{D^{ik}\right\} = \begin{Bmatrix} u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}$$

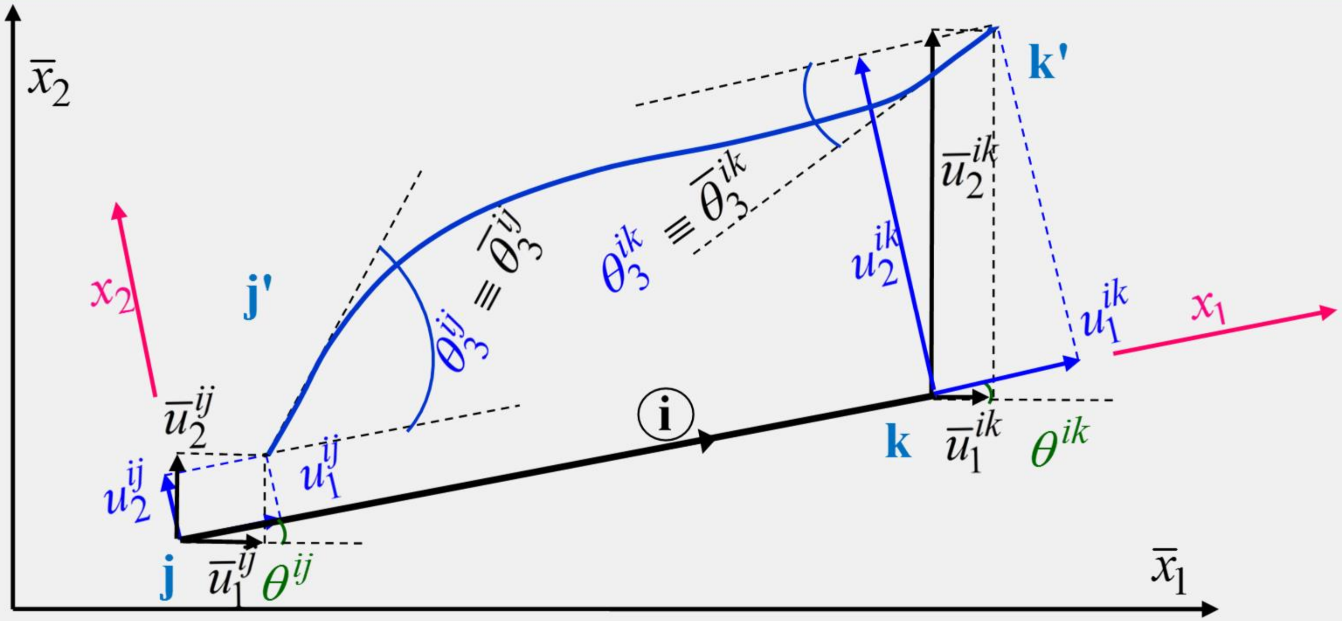
[illegible] *k*

$$\left\{D^{i}\right\} = \begin{bmatrix} \left\{D^{ij}\right\} \\ ---- \\ \left\{D^{ik}\right\} \end{bmatrix} = \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ -- \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}$$

[illegible]

$$\left\{\bar{D}^{i}\right\} = \begin{bmatrix} \left\{\bar{D}^{ij}\right\} \\ ---- \\ \left\{\bar{D}^{ik}\right\} \end{bmatrix} = \begin{Bmatrix} \bar{u}_1^{ij} \\ \bar{u}_2^{ij} \\ \bar{\vartheta}_3^{ij} \\ -- \\ \bar{u}_1^{ik} \\ \bar{u}_2^{ik} \\ \bar{\vartheta}_3^{ik} \end{Bmatrix}$$

ƗƚʔյɅɇȼƗȺƚƗ Ɨĵ ȼʐյɇȾƗ ƗɇȾɨ ȺƚȾƚ ȺɄƚɄȺɋȾƚ Ʉȿյ ƚƗƚȾƚ



ȚɎɗŮůɗɛŮɞ Űɞg Ɏɜɞɜ Ű Űɞg ɛɏəɞggŮ á Űɞg ´ əŰɨůɞg ɗŰ ɏɢŮɐ
Ű) Űɜɞɜɚɐ ´ Űɉ Űɛɧɉ űɤ ůɗ

$$u(x) = u_1^{ij}\psi_1(x) + u_1^{ik}\psi_4(x)$$

ɓ) Ůɓəɏɉ ůɨŰ ´ Űɉ Űɛɧɉ űɤ ůɗ

$$\mathrm{v}(x) = u_2^{ij}\psi_2(x) + \vartheta_3^{ij}\psi_3(x) + u_2^{ik}\psi_5(x) + \vartheta_3^{ik}\psi_6(x)$$

[illegible]

[illegible]) $u(x) = u_1^{ij}\psi_1(x) + u_1^{ik}\psi_4(x)$

$\psi_1(x),\ \psi_4(x) \rightarrow$ [illegible] $\rightarrow$ [illegible] $u_1^{ij} = 1, u_1^{ik} = 0$ [illegible] $u_1^{ij} = 0, u_1^{ik} = 1$, [illegible].

[illegible]

$$\psi_1\left(x\right) = 1 - \xi,\ \xi = x/L$$

$$\psi_4\left(x\right) = \xi,\ \xi = x/L$$

[illegible]

b) [illegible]

$$\mathrm{v}(x) = u_2^{ij}\psi_2(x) + \vartheta_3^{ij}\psi_3(x) + u_2^{ik}\psi_5(x) + \vartheta_3^{ik}\psi_6(x)$$

$\psi_2(x),\ \psi_3(x),\ \psi_5(x),\ \psi_6(x)$ $\rightarrow$ [illegible] $\rightarrow$ [illegible]

- $u_2^{ij} = 1$ [illegible] $\vartheta_3^{ij} = u_2^{ik} = \vartheta_3^{ik} = 0$
- $\vartheta_3^{ij} = 1$ [illegible] $u_2^{ij} = u_2^{ik} = \vartheta_3^{ik} = 0$
- $u_2^{ik} = 1$ [illegible] $u_2^{ij} = \vartheta_3^{ij} = \vartheta_3^{ik} = 0$
- $\vartheta_3^{ik} = 1$ [illegible] $u_2^{ij} = \vartheta_3^{ij} = u_2^{ik} = 0$, [illegible].

[illegible]

b) [illegible]

[illegible] (... Bernoulli)

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\psi_i(x)}{dx^2}\right]=0$$

[illegible] → [illegible]

[illegible]:

$$\psi_i(x)=c_1\frac{x^3}{3}+c_2\frac{x^2}{2}+c_3x+c_4 \qquad i=2,3,5,6$$

$$\psi_i{}'(x)=c_1\frac{x^2}{2}+c_2x+c_3$$

[illegible] c_1, c_2, c_3, c_4 [illegible]

ɆɈ?ȷ Λɇȼ ɆȺɫɆ Ɇj ȼƶȷ ɇɃɆ ɆɇɃɉ ȺɫɃɈ ȺɄɫȺɸɃɈ Ʉʂȷ ɫɆɫɃɈ

Ōǖϧ϶ ɔƨǓ Ǘǉʒ ɣ $_2$(x): $\psi_2\left(0\right)=1, \psi_2'\left(0\right)=0, \psi_2\left(L\right)=0, \psi_2'\left(L\right)=0$

$$\psi_2\left(x\right)=1-3\xi^2+2\xi^3,\ \xi=x/L$$

ɔƨǓ Ǘǉʒ ɣ $_3$(x): $\psi_3\left(0\right)=0, \psi_3'\left(0\right)=1, \psi_3\left(L\right)=0, \psi_3'\left(L\right)=0$

$$\psi_3\left(x\right)=L\xi-2\xi^2+\xi^3,\ \xi=x/L$$

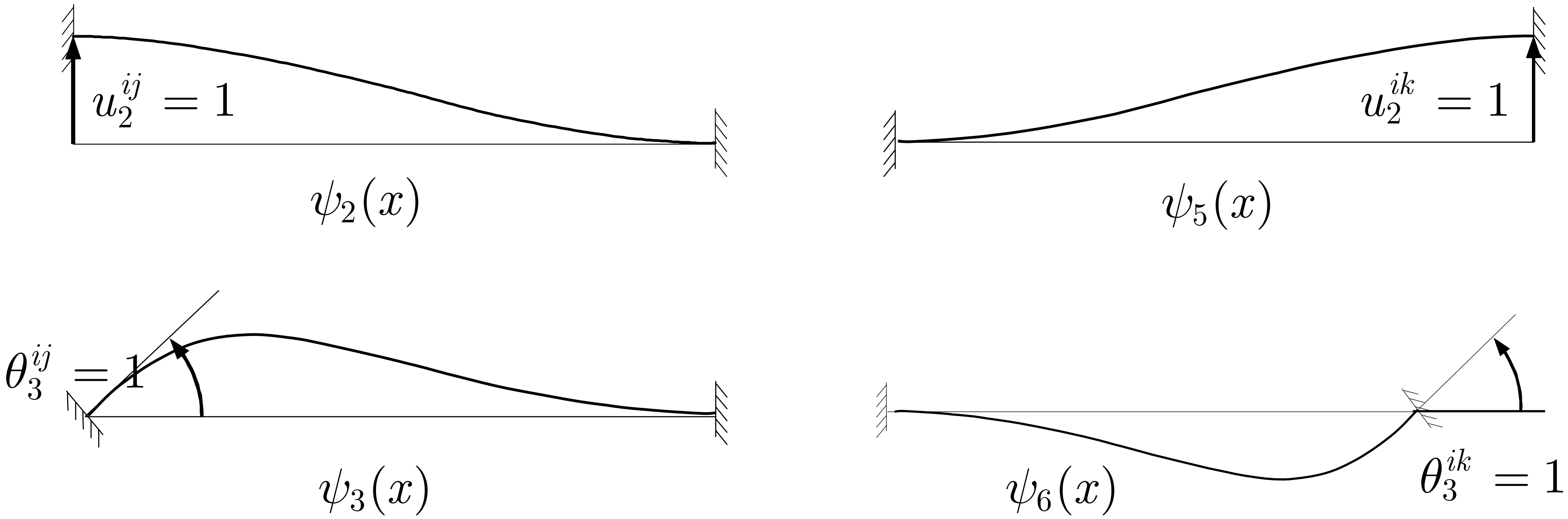
ɔƨǓ Ǘǉʒ ɣ $_5$(x): $\psi_5\left(0\right)=0, \psi_5'\left(0\right)=0, \psi_5\left(L\right)=1, \psi_5'\left(L\right)=0$

$$\psi_5\left(x\right)=3\xi^2-2\xi^3,\ \xi=x/L$$

[illegible]

[illegible] $\psi_6(x)$: $\psi_6\left(0\right)=0, \psi_6'\left(0\right)=0, \psi_6\left(L\right)=0, \psi_6'\left(L\right)=1$

$$\psi_6\left(x\right)=L-\xi^2+\xi^3, \xi=x/L$$



[illegible]

[illegible]	[illegible]	[illegible]
[illegible]	√	✗
[illegible]	√	√

$$\delta W_{\text{in}} = \boxed{\delta W_{\text{in}}^{\ a}} + \boxed{\delta W_{\text{in}}^{\ b}}$$

$\delta W_{\text{in}}^{\ a} \rightarrow$ **[illegible] (axial)**

$\delta W_{\text{in}}^{\ b} \rightarrow$ **[illegible] (bending)**

[illegible]

[illegible]

$$\delta W_{\text{in}}^{a} = \int_V \sigma_x \delta\varepsilon_x^a dV = \int_0^L A(x) E \varepsilon_x^a \delta\varepsilon_x^a dx$$ V: [illegible]

$$\varepsilon_x^a = \frac{\partial u}{\partial x} = u_1^{ij}\psi_1'(x) + u_1^{ik}\psi_4'(x) = $$

$$= \begin{bmatrix}\psi_1'(x) & 0 & 0 & \psi_4'(x) & 0 & 0\end{bmatrix}\begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik}\end{Bmatrix} = \mathbf{N}_a'\mathbf{u}$$

$$\delta\varepsilon_x^a = \mathbf{N}_a'\delta\mathbf{u}$$

[illegible] $$\mathbf{N}_a = \begin{bmatrix}\psi_1(x) & 0 & 0 & \psi_4(x) & 0 & 0\end{bmatrix}$$

[illegible] $$\mathbf{u} = \begin{Bmatrix} u_1^{ij} & u_2^{ij} & \vartheta_3^{ij} & u_1^{ik} & u_2^{ik} & \vartheta_3^{ik}\end{Bmatrix}^T$$

[illegible]

[illegible]

$$\delta W_{\text{in}}^{\ a} = \delta \mathbf{u}^T \left(\int_0^L EA(x) \mathbf{N}_a'^T \mathbf{N}_a' dx \right) \mathbf{u} \qquad \textbf{(1)}$$

[illegible]:

$$\delta W_{\text{in}}^{\ b} = \int_V \sigma_x \delta \varepsilon_x^b dx$$

$$\sigma_x = \frac{M}{I} y = -Eyu'' =$$

$$= -Ey[0 \quad \psi_2''(x) \quad \psi_3''(x) \quad 0 \quad \psi_5''(x) \quad \psi_6''(x)] \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix} = -Ey\mathbf{N}_b''\mathbf{u}$$

$$\delta \varepsilon_x^b = -y \mathbf{N}_b'' \delta \mathbf{u}$$

[illegible]

[illegible] $\mathbf{N}_b = \begin{bmatrix} 0 & \psi_2(x) & \psi_3(x) & 0 & \psi_5(x) & \psi_6(x) \end{bmatrix}$

[illegible] $\mathbf{u} = \begin{Bmatrix} u_1^{ij} & u_2^{ij} & \vartheta_3^{ij} & u_1^{ik} & u_2^{ik} & \vartheta_3^{ik} \end{Bmatrix}^T$

[illegible]

$$\delta W_{\text{in}}^{\ \ b} = \delta \mathbf{u}^T \left(\int_0^L EI(x) \mathbf{N}_b''^{\ T} \mathbf{N}_b'' dx \right) \mathbf{u} \qquad \textbf{(2)}$$

(1) + (2):

$$\delta W_{\text{in}} = \delta W_{\text{in}}^{\ \ a} + \delta W_{\text{in}}^{\ \ b} = \delta \mathbf{u}^T \left(\int_0^L EA(x) \mathbf{N}_a'^{T} \mathbf{N}_a' dx + \int_0^L EI(x) \mathbf{N}_b''^{\ T} \mathbf{N}_b'' dx \right) \mathbf{u}$$

ȼ ȷ Λɉ ȼ ϙɟ?ȷ ɇϙ? ȺΛɑ̃ϙ? ɑ̃ɥ Ɇɇƀɥ ȺɫƁ ȺɄɄȺϙƀɈ Ʉʂȷ ɆɫƀɈ

ɑ̃ʂŬ Ǘɞ ɏɟ ϽϬ Ǘɍ з Ůɞɣ Ǘɟ ɘɩ з ŭɡзϒɛ Ǘɍ з ɘůɢɨ Ǘa

$$\delta W_{ex} = \delta \mathbf{u}^T \mathbf{f} \quad \text{ɧ´ ϭɡ} \quad \mathbf{f}^T = \begin{bmatrix} F_1^{ij} & F_2^{ij} & M_3^{ij} & F_1^{ik} & F_2^{ik} & M_3^{ik} \end{bmatrix}$$

Ⱥзɘůɩ зϭзǗɠ ($\delta W_{in} = \delta W_{ex}$):

$$\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} = \begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\ k_{41} & 0 & 0 & k_{44} & 0 & 0 \\ 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\ 0 & k_{62} & k_{63} & 0 & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}$$

$$\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} = \begin{bmatrix} \boxed{k_{11}} & 0 & 0 & \boxed{k_{14}} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\ \boxed{k_{41}} & 0 & 0 & \boxed{k_{44}} & 0 & 0 \\ 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\ 0 & k_{62} & k_{63} & 0 & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}$$

$$\boxed{k_{ij} = \int_0^L EA\left(x\right)\psi_i'\left(x\right)\psi_j'\left(x\right)dx} \qquad i,j = 1,4$$

[illegible]

$$
\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} =
\begin{bmatrix}
k_{11} & 0 & 0 & k_{14} & 0 & 0 \\
0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\
0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\
k_{41} & 0 & 0 & k_{44} & 0 & 0 \\
0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\
0 & k_{62} & k_{63} & 0 & k_{65} & k_{66}
\end{bmatrix}
\begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}
$$

$$
k_{ij} = \int_0^L EI\left(x\right)\psi_i''\left(x\right)\psi_j''\left(x\right)dx \qquad i, j = 2, 3, 5, 6
$$

ȼ ȷ Λɉ ȼ ɋɟ?ȷ ɇɋ? ȺΛũɋ? ũɏ ɆɇɃɏ ȺɃ ȺɄɄȺɋɃɈ Ʉşȷ ɆɃɈ

$$
\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} =
\begin{bmatrix}
k_{11} & 0 & 0 & k_{14} & 0 & 0 \\
0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\
0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\
k_{41} & 0 & 0 & k_{44} & 0 & 0 \\
0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\
0 & k_{62} & k_{63} & 0 & k_{65} & k_{66}
\end{bmatrix}
\begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}
$$

$$\left\{ A^i \right\}$$

ȼ յ Λɉ ȼ ɸɟ?յ ɇq? ΆΛũq? ũɨɉ ɆɇɃɨɉ ΆɇɃ ΆɄɄΆɸɃɟ Ʉʂյ ɆΈɃɟ

$$
\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix}
=
\begin{bmatrix}
k_{11} & 0 & 0 & k_{14} & 0 & 0 \\
0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\
0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\
k_{41} & 0 & 0 & k_{44} & 0 & 0 \\
0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\
0 & k_{62} & k_{63} & 0 & k_{65} & k_{66}
\end{bmatrix}
\begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}
$$

$$\left\{ A^i \right\} \qquad \left[k^i \right]$$

ȼ ȷ Λɉ ȼ ȹɁ?ȷ ȼɋ? ȺΛũɋ? ũɈ ɆȼȾɈ ȺȾ
ȺɄɄȺɋȾɁ Ʉȿȷ ȽȾɁ

$$
\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} =
\begin{bmatrix}
k_{11} & 0 & 0 & k_{14} & 0 & 0 \\
0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\
0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\
k_{41} & 0 & 0 & k_{44} & 0 & 0 \\
0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\
0 & k_{62} & k_{63} & 0 & k_{65} & k_{66}
\end{bmatrix}
\begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}
$$

$$\left\{ A^i \right\} \qquad \left[k^i \right] \qquad \left\{ D^i \right\}$$

ȼ ȷ Ʌɉ ȼ ɋɊ?ȷ ɇɋ? ȺɅũɋ? ũɊ ȵɇȸɊ ȺɊȸ ȺɄɄȺɋȸɊ Ʉʂȷ ɄȵɊȸɊ

ȼŮƅɵɤ,

$$\begin{Bmatrix} F_1^{ij} \\ F_2^{ij} \\ M_3^{ij} \\ F_1^{ik} \\ F_2^{ik} \\ M_3^{ik} \end{Bmatrix} = \begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\ k_{41} & 0 & 0 & k_{44} & 0 & 0 \\ 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\ 0 & k_{62} & k_{63} & 0 & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1^{ij} \\ u_2^{ij} \\ \vartheta_3^{ij} \\ u_1^{ik} \\ u_2^{ik} \\ \vartheta_3^{ik} \end{Bmatrix}$$

⬇

$$\left\{ A^i \right\} = \left[k^i \right] \left\{ D^i \right\}$$